

# IDEAL-RELATED $K$ -THEORY FOR LEAVITT PATH ALGEBRAS AND GRAPH $C^*$ -ALGEBRAS

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**ABSTRACT.** We introduce a notion of ideal-related  $K$ -theory for rings, and use it to prove that if two complex Leavitt path algebras  $L_{\mathbb{C}}(E)$  and  $L_{\mathbb{C}}(F)$  are Morita equivalent (respectively, isomorphic), then the ideal-related  $K$ -theories (respectively, the unital ideal-related  $K$ -theories) of the corresponding graph  $C^*$ -algebras  $C^*(E)$  and  $C^*(F)$  are isomorphic. This has consequences for the “Morita equivalence conjecture” and “isomorphism conjecture” for graph algebras, and allows us to prove that when  $E$  and  $F$  belong to specific collections of graphs whose  $C^*$ -algebras are classified by ideal-related  $K$ -theory, Morita equivalence (respectively, isomorphism) of the Leavitt path algebras  $L_{\mathbb{C}}(E)$  and  $L_{\mathbb{C}}(F)$  implies strong Morita equivalence (respectively, isomorphism) of the graph  $C^*$ -algebras  $C^*(E)$  and  $C^*(F)$ . We state a number of corollaries that describe various classes of graphs where these implications hold. In addition, we conclude with a classification of Leavitt path algebras of amplified graphs similar to the existing classification for graph  $C^*$ -algebras of amplified graphs.

## 1. INTRODUCTION

In [3] Gene Abrams and the second named author examined the relationship between the structures of the Leavitt path algebra and the graph  $C^*$ -algebra of a given graph. In [3, §9] the authors made two conjectures: The *Morita equivalence conjecture for graph algebras* states that if  $E$  and  $F$  are graphs and  $L_{\mathbb{C}}(E)$  is Morita equivalent to  $L_{\mathbb{C}}(F)$ , then  $C^*(E)$  is strongly Morita equivalent to  $C^*(F)$ . The *isomorphism conjecture for graph algebras* states that if  $E$  and  $F$  are graphs and  $L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$  (as rings), then  $C^*(E) \cong C^*(F)$  (as  $*$ -algebras).

In [3] the authors showed that the Morita equivalence conjecture and the isomorphism conjecture hold for row-finite graphs whose associated  $C^*$ -algebras are simple. This was accomplished by arguing that if  $L_{\mathbb{C}}(E)$  is Morita equivalent to  $L_{\mathbb{C}}(F)$ , then the algebraic  $K$ -theories of  $L_{\mathbb{C}}(E)$  and  $L_{\mathbb{C}}(F)$  are isomorphic, which implies the topological  $K$ -theories of  $C^*(E)$

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*Date:* July 25, 2012.

*2000 Mathematics Subject Classification.* Primary: 46L35, 16D70.

*Key words and phrases.* graph  $C^*$ -algebras, Leavitt path algebras, graph algebras, classification,  $K$ -theory.

This work was partially supported by a grant from the Simons Foundation (#210035 to Mark Tomforde).

and  $C^*(F)$  are isomorphic, and then classification theorems for simple  $C^*$ -algebras imply that  $C^*(E)$  and  $C^*(F)$  are strongly Morita equivalent. A similar argument can be made using isomorphism in place of Morita equivalence by keeping track of the position of the class of the unit in the  $K$ -group. The nontrivial parts of this argument involve (1) showing that the algebraic  $K$ -theories of  $L_{\mathbb{C}}(E)$  and  $L_{\mathbb{C}}(F)$  are isomorphic implies the topological  $K$ -theories of  $C^*(E)$  and  $C^*(F)$  are isomorphic, and (2) applying existing classification theorems for simple  $C^*$ -algebras.

Recently, classification theory for  $C^*$ -algebras has made a number of accomplishments in classifying particular collections of nonsimple  $C^*$ -algebras, and graph  $C^*$ -algebras have provided a fertile testing grounds in which many of these theorems may be applied. Indeed, there is now a large list of distinct classes of graph  $C^*$ -algebras that are classified by  $K$ -theoretic information. In all cases, the classifying invariant has been ideal-related  $K$ -theory, which consists of all cyclic six-term exact sequences of (topological)  $K$ -groups for all subquotients and the natural transformations between them. (In the literature, ideal-related  $K$ -theory is also known by a variety of other names, including “filtered  $K$ -theory” and “filtrated  $K$ -theory”.)

The purpose of this article is to define a notion of ideal-related algebraic  $K$ -theory for rings that applies to Leavitt path algebras, and to show that if two Leavitt path algebras of graphs satisfying Condition (K) have isomorphic ideal-related algebraic  $K$ -theories, then the corresponding graph  $C^*$ -algebras have isomorphic ideal-related topological  $K$ -theories. This result allows one to verify both the Morita equivalence conjecture and the isomorphism conjecture for graphs whose  $C^*$ -algebras are classified by ideal-related topological  $K$ -theory, and we describe many of these specific classes in a number of corollaries to our main results. Throughout, we need the graphs to satisfy Condition (K) to ensure, among other things, that the ideals in the graph algebra correspond to saturated hereditary subsets in the graph. (We mention that at the time this paper is written, all existing classification results for graph  $C^*$ -algebras require Condition (K) as well, so this is a mild hypothesis in the context of the existing theory.)

In addition to verifying the Morita equivalence conjecture and isomorphism conjecture for specific classes of graphs, our results on ideal-related  $K$ -theory also allow us to give a classification of Leavitt path algebras of amplified graphs with finitely many vertices that is similar to the existing classification for  $C^*$ -algebras. Specifically, we show that two such Leavitt path algebras are classified by their ideal-related  $K$ -theories and also by the transitive closures of their graphs. This is exactly what happens in the graph  $C^*$ -algebra case, and hence gives us a converse to the isomorphism conjecture for amplified graphs with finitely many vertices: If  $E$  and  $F$  are amplified graphs with a finite number of vertices, then  $L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$  (as rings) if and only if  $C^*(E) \cong C^*(F)$  (as  $*$ -algebras).

This paper is organized as follows. In Section 2 we discuss notation and preliminaries regarding graphs, graph algebras, and algebraic  $K$ -theory of

rings. In Section 3 we introduce a definition for the ideal-related algebraic  $K$ -theory of a ring. We face some obstacles that do not occur in the  $C^*$ -algebra setting: first of all, we need to consider all  $K$ -groups (not just the  $K_0$ -group and  $K_1$ -group) due to lack of Bott periodicity, and second we need to require that the ring and all of its subquotients satisfy *excision* in order to ensure that long exact sequences of the  $K$ -groups exist. We conclude this section by showing that Leavitt path algebras have the property that all of their subquotients satisfy excision, and thus our definition of ideal-related algebraic  $K$ -theory applies to them. In Section 4 we show in Theorem 4.9 that if the Leavitt path algebras of two graphs satisfying Condition (K) have isomorphic ideal-related algebraic  $K$ -theories, then the  $C^*$ -algebras of those graphs have isomorphic ideal-related topological  $K$ -theories. Using this result, we are able to show in Theorem 4.10 that the Morita equivalence conjecture for graph algebras holds for any class of graphs whose  $C^*$ -algebras are classified up to strong Morita equivalence (equivalently, stable isomorphism) by their ideal-related topological  $K$ -theory. By applying existing classification results, we are then able to establish the Morita equivalence conjecture for many specific classes of graphs, and in Corollary 4.12 through Corollary 4.18 we do so for a number of classes. In Section 5 we consider the isomorphism conjecture for graph algebras, which requires us to examine the position of the class of the unit in  $K$ -theory. We show in Theorem 5.11 that if the ideal-related algebraic  $K$ -theories of two unital Leavitt path algebras of graphs satisfying Condition (K) are isomorphic via an isomorphism taking the class of the unit to the class of the unit, then there is an isomorphism between the ideal-related topological  $K$ -theories of the graph  $C^*$ -algebras taking the class of the unit to the class of the unit. Using this, we are able to show in Theorem 5.12 that the isomorphism conjecture for graph algebras holds for any class of graphs whose  $C^*$ -algebras are classified up to isomorphism by their ideal-related topological  $K$ -theory plus the position of the class of the unit. By applying existing classification results, we are then able to establish the isomorphism conjecture for many specific classes of graphs in Corollary 5.13 through Corollary 5.15. In Section 6 we use our results on ideal-related  $K$ -theory to prove a classification result for amplified graphs. In Theorem 6.7 we show that the (complex) Leavitt path algebras of two amplified graphs with finitely many vertices are isomorphic if and only if the  $C^*$ -algebras of those graphs are isomorphic if and only if the transitive closures of the graphs are isomorphic.

## 2. NOTATION AND PRELIMINARIES

All of the graphs that we consider will be directed graphs with countably many edges and vertices. We need this countability hypothesis to ensure that all our graph  $C^*$ -algebras are separable, which is a necessary hypothesis for many of the classification results applied. The countability hypothesis also ensures that the Leavitt path algebras of our graphs have countable

sets of enough idempotents, which allows us to use the fact that  $L_K(E)$  is Morita equivalent to  $L_K(F)$  if and only if  $M_\infty(L_K(E)) \cong M_\infty(L_K(F))$  (see [3, Definition 9.6, Definition 9.9, and Proposition 9.10]).

**Definition 2.1.** A *directed graph*  $E := (E^0, E^1, r_E, s_E)$  consists of a countable set of vertices  $E^0$ , a countable set of edges  $E^1$ , and maps  $r_E : E^1 \rightarrow E^0$  and  $s_E : E^1 \rightarrow E^0$  identifying the range and source of each edge. When we have only one graph, or it is clear from context, we will often drop the subscript on the range and source maps and simply write  $r := r_E$  or  $s := s_E$ .

**Definition 2.2.** Let  $E = (E^0, E^1, r_E, s_E)$  be a graph. A *path* of  $E$  is a finite sequence of edges  $\alpha := e_1 e_2 \dots e_n$  with  $r_E(e_i) = s_E(e_{i+1})$  for  $1 \leq i \leq n$ , and we call the natural number  $n$  the *length* of the graph. We also consider the vertices to be paths of length zero. We extend the range and source maps to the collection of paths as follows: If  $\alpha = e_1 e_2 \dots e_n$  is a path of positive length, we set  $s_E(\alpha) = s_E(e_1)$  and  $r_E(\alpha) = r_E(e_n)$ , and if  $\alpha = v \in E^0$  is a path of length zero we set  $s_E(v) = r_E(v) = v$ .

A *cycle* in  $E$  is a path  $\alpha := e_1 \dots e_n$  of positive length with  $s_E(e_1) = r_E(e_n)$ . We call the vertex  $s_E(e_1)$  the *base point* of the cycle  $\alpha$ . A cycle is called a *simple cycle* if  $r_E(e_i) \neq s_E(e_1)$  for  $1 \leq i \leq n-1$ . A graph  $E$  is said to satisfy *Condition (K)* if no vertex of  $E$  is the base point of exactly one simple cycle.

A graph is called an *amplified graph* if for every two vertices  $v, w \in E^0$  there are either no edges from  $v$  to  $w$  or there are countably many edges from  $v$  to  $w$ . (Equivalently, a graph is an amplified graph if whenever there is an edge between one vertex  $v$  to another vertex  $w$ , then there are infinitely many edges from  $v$  to  $w$ .)

**Definition 2.3.** Let  $E = (E^0, E^1, r_E, s_E)$  be a graph and let  $R$  be a ring. A collection  $\{v, e, e^* : v \in E^0, e \in E^1\} \subseteq R$  is a *Leavitt  $E$ -family in  $R$*  if  $\{v : v \in E^0\}$  is a collection of pairwise orthogonal idempotents and the following conditions are satisfied:

- (1)  $s_E(e)e = er_E(e) = e$  for all  $e \in E^1$ ;
- (2)  $r_E(e)e^* = e^*s_E(e) = e^*$  for all  $e \in E^1$ ;
- (3)  $e^*f = \delta_{e,f}r_E(e)$  for all  $e, f \in E^1$ ; and
- (4)  $v = \sum_{e \in s_E^{-1}(\{v\})} ee^*$  whenever  $0 < |s_E^{-1}(\{v\})| < \infty$ .

If  $K$  is a field, the *Leavitt path algebra of  $E$  with coefficients in  $K$* , denoted by  $L_K(E)$ , is the universal  $K$ -algebra generated by a Leavitt  $E$ -family.

**Definition 2.4.** Let  $E = (E^0, E^1, r_E, s_E)$  be a graph. A collection

$$\{p_v, s_e : v \in E^0, e \in E^1\}$$

in a  $C^*$ -algebra  $\mathfrak{A}$  is a *Cuntz-Krieger  $E$ -family in  $\mathfrak{A}$*  if  $\{p_v : v \in E^0\}$  consists of pairwise orthogonal projections,  $\{s_e : v \in E^0\}$  is a collection of partial isometries, and the following conditions are satisfied:

- (CK1)  $s_e^* s_f = \delta_{e,f} p_{r_E(e)}$  for all  $e, f \in E^1$ ;

- (CK2)  $s_e s_e^* \leq p_{s_E(e)}$  for all  $e \in E^1$ ; and  
 (CK3)  $p_v = \sum_{e \in s_E^{-1}(\{v\})} s_e s_e^*$  whenever  $0 < |s_E^{-1}(\{v\})| < \infty$ .

The graph  $C^*$ -algebra, denoted  $C^*(E)$ , is defined to be the universal  $C^*$ -algebra generated by a Cuntz-Krieger  $E$ -family.

**Definition 2.5.** Let  $E$  be a graph and let  $v, w$  be in  $E^0$ . We write  $v \geq w$  if there exists a path from  $v$  to  $w$ . A subset  $H$  of  $E^0$  is *hereditary* if for each  $v, w \in E^0$  with  $v \geq w$ ,  $v \in H$  implies that  $w \in H$ . A hereditary subset  $H$  of  $E^0$  is *saturated* if for each  $v \in E^0$  with  $0 < |s_E^{-1}(\{v\})| < \infty$ , we have

$$r_E(s_E^{-1}(\{v\})) \subseteq H \text{ implies } v \in H.$$

We will denote the lattice of saturated hereditary subsets of  $E^0$  by  $\mathcal{H}(E)$ . Note that we always have  $\emptyset \in \mathcal{H}(E)$  and  $E^0 \in \mathcal{H}(E)$ .

**Notation 2.6.** Let  $E$  be a graph and let  $H$  be an element of  $\mathcal{H}(E)$ . We write  $I_H^{\text{alg}}$  to denote the two-sided ideal of  $L_{\mathbb{C}}(E)$  generated by  $\{v : v \in H\}$ , and we write  $I_H^{\text{top}}$  to denote the closed two-sided ideal of  $C^*(E)$  generated by  $\{p_v : v \in H\}$ .

We define a conjugate linear involution on  $L_{\mathbb{C}}(E)$  by

$$\left( \sum \lambda_i \alpha_i \beta_i^* \right)^* = \sum \overline{\lambda_i} \beta_i \alpha_i^*.$$

With this involution  $L_{\mathbb{C}}(E)$  is a complex  $*$ -algebra with the universal property that if  $\mathfrak{A}$  is a complex  $*$ -algebra and  $\{a_v, b_e : v \in E^0, e \in E^1\} \subseteq \mathfrak{A}$  is a set of elements satisfying

- (1) the  $a_v$ 's are pairwise orthogonal and  $a_v = a_v^2 = a_v^*$  for all  $v \in E^0$ ;
- (2)  $a_{s_E(e)} b_e = b_e a_{r_E(e)} = b_e$  for all  $e \in E^1$ ;
- (3)  $b_e^* b_f = \delta_{e,f} a_{r_E(e)}$  for all  $e, f \in E^1$ ; and
- (4)  $a_v = \sum_{e \in s_E^{-1}(\{v\})} b_e b_e^*$  whenever  $0 < |s_E^{-1}(\{v\})| < \infty$ ,

then there exists a unique algebra  $*$ -homomorphism  $\phi : L_{\mathbb{C}}(E) \rightarrow \mathfrak{A}$  satisfying  $\phi(v) = a_v$  and  $\phi(e) = b_e$  for all  $v \in E^0$  and  $e \in E^1$ .

**Theorem 2.7.** [30, Theorem 7.3]

- (1) For any graph  $E$ , there exists an injective algebra  $*$ -homomorphism

$$\iota_E : L_{\mathbb{C}}(E) \rightarrow C^*(E)$$

with  $\iota_E(v) = p_v$  and  $\iota_E(e) = s_e$  for all  $v \in E^0$  and  $e \in E^1$ .

- (2) If  $E$  is a row-finite graph satisfying Condition (K), then the map  $I_H^{\text{alg}} \mapsto I_H^{\text{top}}$  is a lattice isomorphism from the lattice of two-sided ideals of  $L_{\mathbb{C}}(E)$  onto the lattice of closed two-sided ideals of  $C^*(E)$ . Moreover, the closure of  $\iota_E(I_H^{\text{alg}})$  is equal to  $I_H^{\text{top}}$  for all  $H \in \mathcal{H}(E)$ .

**Remark 2.8.** In light of Theorem 2.7, we shall use the map  $\iota_E : L_{\mathbb{C}}(E) \rightarrow C^*(E)$  to identify the set of generators  $\{v, e, e^* : v \in E^0, e \in E^1\} \subseteq L_{\mathbb{C}}(E)$  with the set of generators  $\{p_v, s_e, s_e^* : v \in E^0, e \in E^1\} \subseteq C^*(E)$ , and write  $L_{\mathbb{C}}(E) \subseteq C^*(E)$  when we do so.

**Definition 2.9.** For every  $H \in \mathcal{H}(E)$ , define  $\iota_{E,H} : I_H^{\text{alg}} \rightarrow I_H^{\text{top}}$  by  $\iota_{E,H} = (\iota_E)|_H$ .

Note that if  $H_1, H_2 \in \mathcal{H}(E)$  with  $H_1 \subseteq H_2$ , then the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & I_{H_1}^{\text{alg}} & \longrightarrow & I_{H_2}^{\text{alg}} & \longrightarrow & I_{H_2}^{\text{alg}}/I_{H_1}^{\text{alg}} \longrightarrow 0 \\ & & \downarrow \iota_{E,H_1} & & \downarrow \iota_{E,H_2} & & \downarrow \iota_{E,H_2/H_1} \\ 0 & \longrightarrow & I_{H_1}^{\text{top}} & \longrightarrow & I_{H_2}^{\text{top}} & \longrightarrow & I_{H_2}^{\text{top}}/I_{H_1}^{\text{top}} \longrightarrow 0 \end{array}$$

commutes.

For a ring  $R$  we let  $K_*^{\text{alg}}(R)$  denote the algebraic  $K$ -theory of a ring,  $KH_*(R)$  denote the homotopy algebraic  $K$ -theory introduced by C. Weibel in [31]. Recall that there is a comparison map  $K_*^{\text{alg}}(R) \rightarrow KH_*(R)$  (see [31] for details). For a Banach algebra  $\mathfrak{A}$  we let  $K_*^{\text{top}}(\mathfrak{A})$  denote the topological  $K$ -theory of  $\mathfrak{A}$ .

**Definition 2.10.** Let  $H_1, H_2 \in \mathcal{H}(E)$  with  $H_1 \subseteq H_2$ . We define the comparison map  $\gamma_{n,H_2/H_1}^E : K_n^{\text{alg}}(I_{H_2}^{\text{alg}}/I_{H_1}^{\text{alg}}) \rightarrow K_n^{\text{top}}(I_{H_2}^{\text{top}}/I_{H_1}^{\text{top}})$  to be the composition

$$K_n^{\text{alg}}(I_{H_2}^{\text{alg}}/I_{H_1}^{\text{alg}}) \rightarrow K_n^{\text{alg}}(I_{H_2}^{\text{top}}/I_{H_1}^{\text{top}}) \rightarrow KH_n(I_{H_2}^{\text{top}}/I_{H_1}^{\text{top}}) \rightarrow K_n^{\text{top}}(I_{H_2}^{\text{top}}/I_{H_1}^{\text{top}}).$$

When  $H_1 = \emptyset$  we write  $\gamma_{n,H_2}^E := \gamma_{n,H_2/H_1}^{\mathbb{C}}$ . When  $H_2 = E^0$  and  $H_1 = \emptyset$ , we write  $\gamma_n^E := \gamma_{n,H_2/H_1}^E$ .

**Remark 2.11.** Recall that if  $\mathfrak{A}$  is a unital  $C^*$ -algebra, then there exists a surjective homomorphism

$$K_1^{\text{alg}}(\mathfrak{A}) = \text{GL}(\mathfrak{A})/[\text{GL}(\mathfrak{A}), \text{GL}(\mathfrak{A})] \rightarrow \text{GL}(\mathfrak{A})/\text{GL}(\mathfrak{A})_0 = K_1^{\text{top}}(\mathfrak{A}).$$

Let  $\mathfrak{A}$  be a  $C^*$ -algebra. Denote the ring obtained by adjoining a unit to  $\mathfrak{A}$  by  $\tilde{\mathfrak{A}}^{\text{alg}}$ , and denote the  $C^*$ -algebra obtained by adjoining a unit to  $\mathfrak{A}$  by  $\tilde{\mathfrak{A}}^{\text{top}}$ . Note that there exists a natural unital ring homomorphism from  $\tilde{\mathfrak{A}}^{\text{alg}}$  to  $\tilde{\mathfrak{A}}^{\text{top}}$  sending  $a + n1 \in \tilde{\mathfrak{A}}^{\text{alg}}$  to  $a + n1 \in \tilde{\mathfrak{A}}^{\text{top}}$ . This homomorphism induces a homomorphism from  $K_1^{\text{alg}}(\tilde{\mathfrak{A}}^{\text{alg}})$  to  $K_1^{\text{top}}(\tilde{\mathfrak{A}}^{\text{top}})$ , and the composition

$$K_1^{\text{alg}}(\mathfrak{A}) \rightarrow KH_1(\mathfrak{A}) \rightarrow K_1^{\text{top}}(\mathfrak{A})$$

is equal to the composition

$$K_1^{\text{alg}}(\mathfrak{A}) \rightarrow K_1^{\text{alg}}(\tilde{\mathfrak{A}}^{\text{alg}}) \rightarrow K_1^{\text{alg}}(\tilde{\mathfrak{A}}^{\text{top}}) \rightarrow K_1^{\text{top}}(\tilde{\mathfrak{A}}^{\text{top}}) = K_1^{\text{top}}(\mathfrak{A}).$$

### 3. IDEAL-RELATED $K$ -THEORY

In this section we recall the definition of ideal-related topological  $K$ -theory for  $C^*$ -algebras, and we introduce a definition of ideal-related algebraic  $K$ -theory for rings.

**Definition 3.1.** If  $R$  is a ring, we let  $\text{Lat}^{\text{alg}}(R)$  denote the lattice of all two-sided ideals of  $R$ . If  $\mathfrak{A}$  is a  $C^*$ -algebra, we let  $\text{Lat}^{\text{top}}(\mathfrak{A})$  denote the lattice of all closed two-sided ideals of  $\mathfrak{A}$ . A *subquotient* of  $R$  is a ring of the form  $I_2/I_1$  for some  $I_2, I_1 \in \text{Lat}^{\text{alg}}(R)$  with  $I_1 \subseteq I_2$ , and a *subquotient* of  $\mathfrak{A}$  is a  $C^*$ -algebra of the form  $\mathfrak{I}_2/\mathfrak{I}_1$  for some  $\mathfrak{I}_2, \mathfrak{I}_1 \in \text{Lat}^{\text{top}}(\mathfrak{A})$  with  $\mathfrak{I}_1 \subseteq \mathfrak{I}_2$ .

**Definition 3.2.** Let  $\mathfrak{A}$  be a  $C^*$ -algebra. Note that for any three ideals  $\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_3 \in \text{Lat}^{\text{top}}(\mathfrak{A})$  with  $\mathfrak{I}_1 \subseteq \mathfrak{I}_2 \subseteq \mathfrak{I}_3$ , the short exact sequence

$$0 \rightarrow \mathfrak{I}_2/\mathfrak{I}_1 \rightarrow \mathfrak{I}_3/\mathfrak{I}_1 \rightarrow \mathfrak{I}_3/\mathfrak{I}_2 \rightarrow 0$$

induces a six-term exact sequence in  $K$ -theory:

$$\begin{array}{ccccc} K_0^{\text{top}}(\mathfrak{I}_2/\mathfrak{I}_1) & \xrightarrow{\iota_*} & K_0^{\text{top}}(\mathfrak{I}_3/\mathfrak{I}_1) & \xrightarrow{\pi_*} & K_0^{\text{top}}(\mathfrak{I}_3/\mathfrak{I}_2) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1^{\text{top}}(\mathfrak{I}_3/\mathfrak{I}_2) & \xleftarrow{\pi_*} & K_1^{\text{top}}(\mathfrak{I}_3/\mathfrak{I}_1) & \xleftarrow{\iota_*} & K_1^{\text{top}}(\mathfrak{I}_2/\mathfrak{I}_1). \end{array}$$

- (1) We define  $K_{\text{ideal}}^{\text{top}}(\mathfrak{A})$  of  $\mathfrak{A}$  to be the collection of all  $K$ -groups thus occurring, equipped with the natural transformations  $\{\iota_*, \pi_*, \partial\}$ .
- (2) The *ideal-related topological  $K$ -theory*  $K_{\text{ideal}}^{\text{top},+}(\mathfrak{A})$  of  $\mathfrak{A}$  is  $K_{\text{ideal}}^{\text{top}}(\mathfrak{A})$  together with the positive cone  $K_0^{\text{top}}(\mathfrak{I}_2/\mathfrak{I}_1)^+$  for all subquotients  $\mathfrak{I}_2/\mathfrak{I}_1$ .

If  $\mathfrak{A}$  and  $\mathfrak{B}$  are  $C^*$ -algebras, we say  $K_{\text{ideal}}^{\text{top},+}(\mathfrak{A})$  is *isomorphic* to  $K_{\text{ideal}}^{\text{top},+}(\mathfrak{B})$ , and write  $K_{\text{ideal}}^{\text{top},+}(\mathfrak{A}) \cong K_{\text{ideal}}^{\text{top},+}(\mathfrak{B})$ , if the following three conditions hold:

- (i) there exists a lattice isomorphism  $\beta : \text{Lat}^{\text{top}}(\mathfrak{A}) \rightarrow \text{Lat}^{\text{top}}(\mathfrak{B})$ ;
- (ii) for each pair of ideals  $\mathfrak{I}_1, \mathfrak{I}_2 \in \text{Lat}^{\text{top}}(\mathfrak{A})$  with  $\mathfrak{I}_1 \subseteq \mathfrak{I}_2$  there exist group isomorphisms

$$\alpha_n^{\mathfrak{I}_1, \mathfrak{I}_2} : K_n^{\text{top}}(\mathfrak{I}_2/\mathfrak{I}_1) \rightarrow K_n^{\text{top}}(\beta(\mathfrak{I}_2)/\beta(\mathfrak{I}_1)) \quad \text{for } n = 0, 1$$

with  $\alpha_0^{\mathfrak{I}_1, \mathfrak{I}_2}$  an order isomorphism; and

- (iii) this collection of isomorphisms preserves all natural transformations.

In this case we consider the pair

$$\phi := \left( \left\{ \alpha_n^{\mathfrak{I}_1, \mathfrak{I}_2} : \mathfrak{I}_1, \mathfrak{I}_2 \in \text{Lat}^{\text{top}}(\mathfrak{A}) \text{ with } \mathfrak{I}_1 \subseteq \mathfrak{I}_2, n = 0, 1 \right\}, \beta \right)$$

to be a morphism from  $K_{\text{ideal}}^{\text{top},+}(\mathfrak{A})$  to  $K_{\text{ideal}}^{\text{top},+}(\mathfrak{B})$ , and we write

$$\phi : K_{\text{ideal}}^{\text{top},+}(\mathfrak{A}) \rightarrow K_{\text{ideal}}^{\text{top},+}(\mathfrak{B})$$

and call  $\phi$  an *isomorphism*. (The only morphisms we will need to consider are isomorphisms.)

**Remark 3.3.** The ideal-related topological  $K$ -theory that we define above is known by several other names in the literature. In addition to “ideal-related  $K$ -theory” it is also called “filtered  $K$ -theory” [24] or “filtrated  $K$ -theory” [20], and it is closely related to the  $K$ -web introduced by Boyle and Huang [11]. To make matters more confusing, Meyer and Nest have

also defined a notion of “filtrated  $K$ -theory” that has additional natural transformations besides the ones we have included here [23, 10]. Although defined differently, it is not known whether the “filtrated  $K$ -theory” of Meyer and Nest is equivalent to the definition we have given above.

In analogy with  $C^*$ -algebras, we wish to introduce a notion of ideal-related  $K$ -theory for rings. In general, algebraic  $K$ -theory does not satisfy Bott periodicity and so we will need all  $K$ -groups — not just  $K_0$  and  $K_1$  — in our definition. In addition, when  $R$  is general ring, and  $I$  is an ideal of  $R$ , the short exact sequence  $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$  does not necessarily induce a long exact sequence of algebraic  $K$ -groups. In order to ensure such long exact sequences for our definition, we must restrict our attention to rings in which every subquotient satisfies excision. We shall show that all Leavitt path algebras have this property, and thus our definition of algebraic ideal-related  $K$ -theory will suffice for our purposes.

**Definition 3.4.** A ring  $R$  is said to satisfy *excision in algebraic  $K$ -theory* if whenever  $A$  is a unital ring that contains  $R$  as a two-sided ideal, the natural map  $K_*^{\text{alg}}(R) \rightarrow K_*^{\text{alg}}(A, R)$  is an isomorphism, where  $K_*^{\text{alg}}(R)$  denotes algebraic  $K$ -theory of  $A$  and  $K_*^{\text{alg}}(A, R)$  denotes the relative algebraic  $K$ -theory of  $A$ . The relative algebraic  $K$ -theory  $K_*^{\text{alg}}(A, R)$  is defined so that the relative algebraic  $K$ -groups satisfy a long exact sequence. Thus if  $R$  satisfies excision in algebraic  $K$ -theory and  $A$  is an arbitrary (not necessarily unital) ring containing  $R$  as a two-sided ideal, then the natural sequence

$$K_n^{\text{alg}}(R) \xrightarrow{\iota_*} K_n^{\text{alg}}(A) \xrightarrow{\pi_*} K_n^{\text{alg}}(A/R) \xrightarrow{\partial_*} K_{n-1}^{\text{alg}}(R)$$

is exact for each  $n \in \mathbb{Z}$ .

**Definition 3.5.** Let  $R$  be a ring such that every subquotient  $I_2/I_1$  of  $R$  satisfies excision in algebraic  $K$ -theory. Then for any three ideals  $I_1, I_2, I_3 \in \text{Lat}^{\text{alg}}(R)$  with  $I_1 \subseteq I_2 \subseteq I_3$ , we have an exact sequence

$$K_n^{\text{alg}}(I_2/I_1) \xrightarrow{\iota_*} K_n^{\text{alg}}(I_3/I_1) \xrightarrow{\pi_*} K_n^{\text{alg}}(I_3/I_2) \xrightarrow{\partial} K_{n-1}^{\text{alg}}(I_3/I_2)$$

for each  $n \in \mathbb{Z}$ .

- (1) We define  $K_{\text{ideal}}^{\text{alg}}(R)$  to be the collection of all  $K$ -groups thus occurring, equipped with the natural transformations  $\{\iota_*, \pi_*, \partial\}$ .
- (2) The *ideal-related algebraic  $K$ -theory*  $K_{\text{ideal}}^{\text{alg},+}(R)$  of  $R$  consists of  $K_{\text{ideal}}^{\text{alg}}(R)$  together with the positive cone of  $K_0^{\text{alg}}(I_2/I_1)$  for all two-sided ideals  $I_2, I_1 \in \text{Lat}^{\text{alg}}(R)$  with  $I_1 \subseteq I_2$ .

Let  $R$  and  $S$  be rings that each have the property that every subquotient satisfies excision in algebraic  $K$ -theory. We will say that  $K_{\text{ideal}}^{\text{alg},+}(R)$  is *isomorphic* to  $K_{\text{ideal}}^{\text{alg},+}(S)$ , and write  $K_{\text{ideal}}^{\text{alg},+}(R) \cong K_{\text{ideal}}^{\text{alg},+}(S)$ , if the following three conditions hold:

- (i) there exists a lattice isomorphism  $\beta : \text{Lat}^{\text{alg}}(R) \rightarrow \text{Lat}^{\text{alg}}(S)$ ;



- (ii) for each pair of ideals  $I_1, I_2 \in \text{Lat}^{\text{alg}}(R)$  with  $I_1 \subseteq I_2$  there exist group isomorphisms

$$\alpha_n^{I_1, I_2} : K_n^{\text{alg}}(I_2/I_1) \rightarrow K_n^{\text{alg}}(\beta(I_2)/\beta(I_1)) \quad \text{for each } n \in \mathbb{Z}$$

with  $\alpha_0^{I_1, I_2}$  an order isomorphism; and

- (iii) this collection of isomorphisms preserves all natural transformations.

In this case we consider the pair

$$\phi := \left( \left\{ \alpha_n^{I_1, I_2} : I_1, I_2 \in \text{Lat}^{\text{alg}}(R) \text{ with } I_1 \subseteq I_2, n \in \mathbb{Z} \right\}, \beta \right)$$

to be a morphism from  $K_{\text{ideal}}^{\text{alg},+}(R)$  to  $K_{\text{ideal}}^{\text{alg},+}(S)$ , and we write

$$\phi : K_{\text{ideal}}^{\text{alg},+}(R) \rightarrow K_{\text{ideal}}^{\text{alg},+}(S)$$

and call  $\phi$  an *isomorphism*. (The only morphisms we will need to consider are isomorphisms.)

**Definition 3.6.** Let  $R$  be a ring. A sequence  $\{e_n\}_{n=1}^{\infty} \subseteq R$  is a *countable approximate unit consisting of idempotents* if the following three properties hold:

- (1) each  $e_n$  is an idempotent,
- (2)  $e_{n+1}e_n = e_n e_{n+1} = e_n$  for all  $n \in \mathbb{N}$ , and
- (3) for any  $r \in R$  there exists  $n \in \mathbb{N}$  such that  $e_n r = r e_n = r$ .

**Remark 3.7.** Note that if  $R$  is a ring with a countable approximate unit consisting of idempotents  $\{e_n\}_{n=1}^{\infty}$ , then  $R = \bigcup_{i=1}^{\infty} e_i R e_i$  with  $e_1 R e_1 \subseteq e_2 R e_2 \subseteq \dots$  and each  $e_n R e_n$  a unital subring of  $R$  with unit  $e_n$ . Thus, in this case,  $R$  is the increasing union of a sequence of unital subrings, and hence also a direct limit of a sequence of unital rings.

**Remark 3.8.** We wish to show that any ring with a countable approximate unit consisting of idempotents satisfies excision in algebraic  $K$ -theory. The most general results on excision are due to Suslin in [28] where he proves that a ring  $R$  satisfies excision if and only if

$$(3.1) \quad \text{Tor}_n^{\tilde{R}}(R, \mathbb{Z}) = 0 \text{ for } n \geq 0,$$

where  $\tilde{R}$  denotes the unitization of  $R$ . A ring  $R$  is said to be  *$H'$ -unital* if it satisfies (3.1). If  $R$  is torsion-free as a  $\mathbb{Z}$ -module, then  $R$  is  *$H'$ -unital* if and only if  $R$  is  *$H$ -unital* in the sense of Wodzicki [32]. Suslin and Wodzicki proved in [29, Theorem B] that if the ring  $R$  is a  $\mathbb{Q}$ -algebra, then  $R$  satisfies excision in algebraic  $K$ -theory if and only if  $R$  is  *$H$ -unital*. Unital rings are both  *$H$ -unital* and  *$H'$ -unital* [4, Remark 2.2]. Using these facts we are able to establish the following lemma.

**Lemma 3.9.** *Any ring with a countable approximate unit consisting of idempotents satisfies excision in algebraic  $K$ -theory.*

*Proof.* Every unital ring is  $H'$ -unital (see [4, Remark 2.2]). In addition, by [4, Remark 2.2], the class of  $H'$ -unital rings is closed under direct limits. Hence all rings with a countable approximate unit consisting of idempotents are  $H'$ -unital (see Remark 3.7). Suslin has proven that a ring satisfies excision in algebraic  $K$ -theory if and only if the ring is  $H'$ -unital [28]. Thus all rings with a countable approximate unit consisting of idempotents satisfy excision in algebraic  $K$ -theory.  $\square$

**Lemma 3.10.** *If  $E$  satisfies Condition (K), then any subquotient of  $L_K(E)$  has a countable approximate unit consisting of idempotents.*

*Proof.* Any Leavitt path algebra has a countable approximate unit consisting of idempotents. Let  $I$  and  $J$  be ideals of  $L_K(E)$  such that  $I \subseteq J$ . By [26, Corollary 6.3],  $J$  is isomorphic to  $L_K(F)$  for some graph  $F$ . Hence,  $J$  has a countable approximate unit consisting of idempotents,  $\{e_n\}_{n=1}^\infty$ . Then  $\{e_n + I\}_{n=1}^\infty$  is a countable approximate unit consisting of idempotents for  $J/I$ .  $\square$

**Remark 3.11.** It follows from Lemma 3.9 and Lemma 3.10 that if  $E$  is a graph satisfying Condition (K), then any subquotient of  $L_K(E)$  satisfies excision in algebraic  $K$ -theory. Thus, the definition of ideal-related algebraic  $K$ -theory applies to  $L_K(E)$  whenever  $E$  satisfies Condition (K).

**Remark 3.12.** If  $E$  is a row-finite graph satisfying Condition (K), then the lattice of ideals of the Leavitt path algebra  $L_K(E)$  is isomorphic to the lattice  $\mathcal{H}(E)$  of saturated hereditary subsets of  $E$ . In this case we may identify any ideal in  $L_K(E)$  with its corresponding saturated hereditary subset of  $E^0$ . If we do this, then whenever  $E_1$  and  $E_2$  are row-finite graphs satisfying Condition (K), an isomorphism between  $K_{\text{ideal}}^{\text{alg}}(L_K(E_1))$  and  $K_{\text{ideal}}^{\text{alg}}(L_K(E_2))$  consists of (i) a lattice isomorphism  $\beta : \mathcal{H}(E_1) \rightarrow \mathcal{H}(E_2)$  together with (ii) group isomorphisms

$$\alpha_n^{H_1, H_2} : K_n^{\text{alg}}(I_{H_2}^{\text{alg}}/I_{H_1}^{\text{alg}}) \rightarrow K_n^{\text{alg}}(I_{\beta(H_2)}^{\text{alg}}/I_{\beta(H_1)}^{\text{alg}}) \quad \text{for each } n \in \mathbb{Z}$$

for each pair  $H_1, H_2 \in \mathcal{H}(E_1)$  with  $H_1 \subseteq H_2$ , that also (iii) preserves all natural transformations. Moreover, this is an isomorphism from  $K_{\text{ideal}}^{\text{alg},+}(L_K(E_1))$  onto  $K_{\text{ideal}}^{\text{alg},+}(L_K(E_2))$  precisely when all the  $\alpha_0^{H_1, H_2}$  are also order isomorphisms.

Similarly, if  $E$  is a graph satisfying Condition (K), then the lattice of closed ideals of the graph  $C^*$ -algebra  $C^*(E)$  is isomorphic to the lattice  $\mathcal{H}(E)$  of saturated hereditary subsets of  $E$ . In this case we may identify any ideal in  $C^*(E)$  with its corresponding saturated hereditary subset of  $E^0$ . If we do this, then whenever  $E_1$  and  $E_2$  are row-finite graphs satisfying Condition (K), an isomorphism between  $K_{\text{ideal}}^{\text{top}}(C^*(E_1))$  and  $K_{\text{ideal}}^{\text{top}}(C^*(E_2))$  consists of (i) a lattice isomorphism  $\beta : \mathcal{H}(E_1) \rightarrow \mathcal{H}(E_2)$  together with (ii) group isomorphisms

$$\alpha_n^{H_1, H_2} : K_n^{\text{top}}(I_{H_2}^{\text{top}}/I_{H_1}^{\text{top}}) \rightarrow K_n^{\text{top}}(I_{\beta(H_2)}^{\text{top}}/I_{\beta(H_1)}^{\text{top}}) \quad \text{for each } n = 0, 1$$

for each pair  $H_1, H_2 \in \mathcal{H}(E_1)$  with  $H_1 \subseteq H_2$ , that also (iii) preserves all natural transformations. Moreover, this is an isomorphism from  $K_{\text{ideal}}^{\text{top},+}(C^*(E_1))$  onto  $K_{\text{ideal}}^{\text{top},+}(C^*(E_2))$  precisely when all the  $\alpha_0^{H_1, H_2}$  are also order isomorphisms.

#### 4. THE MORITA EQUIVALENCE CONJECTURE FOR GRAPH ALGEBRAS

In this section we show that if two complex Leavitt path algebras of graphs satisfying Condition (K) have isomorphic ideal-related algebraic  $K$ -theories, then the corresponding graph  $C^*$ -algebras have isomorphic ideal-related topological  $K$ -theories. This allows us to prove that the Morita equivalence conjecture holds for any class of graphs whose  $C^*$ -algebras are classified by ideal-related topological  $K$ -theory. By applying existing classification theorems, we confirm the Morita equivalence conjecture for a number of specific classes of graphs in the corollaries at the end of this section.

**Definition 4.1.** Let  $M$  be a monoid. A submonoid  $S$  of  $M$  is an *order ideal* of  $M$  if  $x + y \in S$  implies  $x, y \in S$ . Let  $S$  be an order-ideal of  $M$ . Define a congruence  $\sim_S$  on  $M$  by setting  $a \sim_S b$  if and only if there exist  $x, y \in S$  such that  $a + x = b + y$ . Let  $M/S$  be the factor monoid obtained from the congruence.

**Theorem 4.2.** [5, Proposition 1.4] *If  $R$  is an exchange ring and  $I$  is an ideal of  $R$ , then  $V(R/I) \cong V(R)/V(I)$ .*

**Lemma 4.3.** *Let  $E$  be a row-finite graph satisfying Condition (K). If  $H \in \mathcal{H}(E)$ , and  $\overline{E}_{(H, \emptyset)}$  denotes the graph of [26, Definition 4.1], then there exists an algebra isomorphism  $\phi : I_H^{\text{alg}} \rightarrow L\mathbb{C}(\overline{E}_{(H, \emptyset)})$  and a  $*$ -isomorphism  $\overline{\phi} : I_H^{\text{top}} \rightarrow C^*(\overline{E}_{(H, \emptyset)})$  that make the diagram*

$$\begin{array}{ccc} L\mathbb{C}(\overline{E}_{(H, \emptyset)}) & \xrightarrow{\phi} & I_H^{\text{alg}} \\ \downarrow \iota_{\overline{E}_{(H, \emptyset)}} & & \downarrow \iota_{E, H} \\ C^*(\overline{E}_{(H, \emptyset)}) & \xrightarrow{\overline{\phi}} & I_H^{\text{top}} \end{array}$$

*commute. Consequently, we have the following:*

- (1) *For every  $H \in \mathcal{H}(E)$ , the ideal  $I_H^{\text{alg}}$  is an exchange ring.*
- (2) *For every  $H_1, H_2 \in \mathcal{H}(E)$  with  $H_1 \subseteq H_2$ , the natural map from  $V(I_{H_2}^{\text{alg}})$  to  $V(I_{H_2}^{\text{alg}}/I_{H_1}^{\text{alg}})$  is surjective.*

*Proof.* The existence of the isomorphisms and commutativity of the diagram follows from the proofs of [26, Theorem 5.1] and [26, Theorem 6.1]. Statement (1) follows from the fact that  $E$  satisfies Condition (K) implies  $\overline{E}_{(H, \emptyset)}$  satisfies Condition (K), and a graph satisfying Condition (K) implies the associated Leavitt path algebra is an exchange ring [2, Theorem 5.8]. Statement (2) then follows from Theorem 4.2.  $\square$

**Proposition 4.4.** *Let  $E$  be a row-finite graph. Then for every  $H_1, H_2 \in \mathcal{H}(E)$  with  $H_1 \subseteq H_2$ ,*

$$V(\iota_{E, H_2/H_1}) : V(I_{H_2}^{\text{alg}}/I_{H_1}^{\text{alg}}) \rightarrow V(I_{H_2}^{\text{top}}/I_{H_1}^{\text{top}})$$

*is an isomorphism. Consequently,*

$$\gamma_{0, H_2/H_1}^E : K_0^{\text{alg}}(I_{H_2}^{\text{alg}}/I_{H_1}^{\text{alg}}) \rightarrow K_0^{\text{top}}(I_{H_2}^{\text{top}}/I_{H_1}^{\text{top}})$$

*is an order isomorphism.*

*Proof.* We first show the proposition is true for the case that  $H_1 = \emptyset$ . A typical element of  $V(I_{H_2}^{\text{top}})$  has the form  $[p]$  for a projection  $p \in \mathbb{M}_n(I_{H_2}^{\text{top}})$ . By the proof of [6, Theorem 7.1],

$$[p] = [p_{v_1}] + [p_{v_2}] + \cdots + [p_{v_n}]$$

where  $v_i \in E^0$ . Since  $p \in \mathbb{M}_n(I_{H_2}^{\text{top}})$ , and  $V(I_{H_2}^{\text{top}})$  is an order ideal,  $[p_{v_i}] \in V(I_{H_2}^{\text{top}})$  for all  $1 \leq i \leq n$ , and since  $I_{H_2}^{\text{top}}$  is an ideal,  $p_{v_i} \in I_{H_2}^{\text{top}}$  for all  $1 \leq i \leq n$ . Thus,  $v_i \in H_2$  for all  $1 \leq i \leq n$ . Hence,  $[v_1] + [v_2] + \cdots + [v_n] \in V(I_{H_2}^{\text{alg}})$  and

$$V(\iota_E)([v_1] + [v_2] + \cdots + [v_n]) = [p].$$

Therefore,  $V(\iota_{E, H_2})$  is surjective.

Suppose  $x, y \in V(I_{H_2}^{\text{alg}})$  such that  $V(\iota_E)(x) = V(\iota_E)(y)$ . By [6, Theorem 7.1],  $x = y$  in  $V(L_{\mathbb{C}}(E))$ . Hence,  $x = y$  in  $V(I_{H_2}^{\text{alg}})$ . Therefore,  $V(\iota_{E, H_2})$  is injective.

We now deal with the general case. Note that the diagram

$$\begin{array}{ccc} V(I_{H_2}^{\text{alg}}) & \longrightarrow & V(I_{H_2}^{\text{alg}}/I_{H_1}^{\text{alg}}) \\ V(\iota_{E, H_2}) \downarrow & & \downarrow V(\iota_{E, H_2/H_2}) \\ V(I_{H_2}^{\text{top}}) & \longrightarrow & V(I_{H_2}^{\text{top}}/I_{H_1}^{\text{top}}) \end{array}$$

is commutative. By Lemma 4.3 and Theorem 4.2, the horizontal maps are surjective, so that

$$V(I_{H_2}^{\text{alg}}/I_{H_1}^{\text{alg}}) \cong V(I_{H_2}^{\text{alg}})/V(I_{H_1}^{\text{alg}}) \quad \text{and} \quad V(I_{H_2}^{\text{top}}/I_{H_1}^{\text{top}}) \cong V(I_{H_2}^{\text{top}})/V(I_{H_1}^{\text{top}}).$$

Therefore,  $V(\iota_{E, H_2/H_2})$  is surjective. Let  $a, b \in V(I_{H_2}^{\text{alg}})$  and let  $\bar{a}, \bar{b} \in V(I_{H_2}^{\text{alg}}/I_{H_1}^{\text{alg}})$  be the image of  $a$  and  $b$  in  $V(I_{H_2}^{\text{alg}}/I_{H_1}^{\text{alg}})$ , respectively. Suppose  $V(\iota_{E, H_2/H_2})(\bar{a}) = V(\iota_{E, H_2/H_2})(\bar{b})$ . Then there exist  $x, y \in V(I_{H_1}^{\text{top}})$  such that

$$V(\iota_{E, H_2})(a) + x = V(\iota_{E, H_2})(b) + y$$

in  $V(I_{H_2}^{\text{top}})$ . Since  $V(\iota_{E, H_1})$  is an isomorphism, there exist  $c, d \in V(I_{H_1}^{\text{alg}})$  such that  $V(\iota_{E, H_1})(c) = x$  and  $V(\iota_{E, H_1})(d) = y$ . Since  $V(\iota_{E, H_2})$  is an isomorphism, we have that  $a + c = b + d$ . Hence,  $\bar{a} = \bar{b}$ . Therefore,  $V(\iota_{E, H_2/H_2})$  is injective.

The last statement follows from the fact that  $K_0^{\text{alg}}(I_{H_2}^{\text{alg}}/I_{H_1}^{\text{alg}})$  is the enveloping group of  $V(I_{H_2}^{\text{alg}}/I_{H_1}^{\text{alg}})$  and  $K_0(I_{H_2}^{\text{top}}/I_{H_1}^{\text{top}}) = K_0^{\text{top}}(I_{H_2}^{\text{top}}/I_{H_1}^{\text{top}})$  is the enveloping group of  $V(I_{H_2}^{\text{top}}/I_{H_1}^{\text{top}})$ .  $\square$

The following theorem is a result of Ara, Brustenga, and Cortinas in [4].

**Theorem 4.5.** [4, Theorem 7.6 and Theorem 9.1] *Let  $E$  be a row-finite graph and let  $A_E$  be the vertex matrix of  $E$ . Let  $B_E$  be (rectangular) sub-matrix obtained from  $A_E$  by removing the rows corresponding to the sinks of  $E$ . Then*

$$K^{\text{alg}}(L_{\mathbb{C}}(E)) \cong \text{hocofiber}(K^{\text{alg}}(\mathbb{C})^{E^0 \setminus \text{Sink}(E)} \xrightarrow{1-B_E^t} K^{\text{alg}}(\mathbb{C})^{E^0})$$

$$K^{\text{top}}(C^*(E)) \cong \text{hocofiber}(K^{\text{top}}(\mathbb{C})^{E^0 \setminus \text{Sink}(E)} \xrightarrow{1-B_E^t} K^{\text{top}}(\mathbb{C})^{E^0})$$

Consequently, the following chain complexes

$$K_n^{\text{alg}}(\mathbb{C})^{E^0 \setminus \text{Sink}(E)} \xrightarrow{1-B_E^t} K_n^{\text{alg}}(\mathbb{C})^{E^0} \longrightarrow K_n^{\text{alg}}(L_{\mathbb{C}}(E)) \longrightarrow K_{n-1}^{\text{alg}}(\mathbb{C})^{E^0 \setminus \text{Sink}(E)} \xrightarrow{1-B_E^t} K_{n-1}^{\text{alg}}(\mathbb{C})^{E^0}$$

and

$$K_n^{\text{top}}(\mathbb{C})^{E^0 \setminus \text{Sink}(E)} \xrightarrow{1-B_E^t} K_n^{\text{top}}(\mathbb{C})^{E^0} \longrightarrow K_n^{\text{top}}(C^*(E)) \longrightarrow K_{n-1}^{\text{top}}(\mathbb{C})^{E^0 \setminus \text{Sink}(E)} \xrightarrow{1-B_E^t} K_{n-1}^{\text{top}}(\mathbb{C})^{E^0}$$

are exact for all  $n \in \mathbb{Z}$ .

**Lemma 4.6.** *Let  $E$  be a graph and let  $H_1, H_2$ , and  $H_3$  be elements of  $\mathcal{H}(E)$  such that  $H_1 \subseteq H_2 \subseteq H_3$ . Then the diagrams*

$$\begin{array}{ccccccc} K_1^{\text{alg}}(I_{H_2}^{\text{alg}}/I_{H_1}^{\text{alg}}) & \longrightarrow & K_1^{\text{alg}}(I_{H_3}^{\text{alg}}/I_{H_1}^{\text{alg}}) & \longrightarrow & K_1^{\text{alg}}(I_{H_3}^{\text{alg}}/I_{H_2}^{\text{alg}}) & \longrightarrow & K_0^{\text{alg}}(I_{H_2}^{\text{alg}}/I_{H_1}^{\text{alg}}) \\ \gamma_{1,H_2/H_1}^E \downarrow & & \gamma_{1,H_3/H_1}^E \downarrow & & \gamma_{1,H_3/H_2}^E \downarrow & & \gamma_{0,H_2/H_1}^E \downarrow \\ K_1^{\text{top}}(I_{H_2}^{\text{top}}/I_{H_1}^{\text{top}}) & \longrightarrow & K_1^{\text{top}}(I_{H_3}^{\text{top}}/I_{H_1}^{\text{top}}) & \longrightarrow & K_1^{\text{top}}(I_{H_3}^{\text{top}}/I_{H_2}^{\text{top}}) & \longrightarrow & K_0^{\text{top}}(I_{H_2}^{\text{top}}/I_{H_1}^{\text{top}}) \end{array}$$

and

$$\begin{array}{ccccccc} K_0^{\text{alg}}(I_{H_2}^{\text{alg}}/I_{H_1}^{\text{alg}}) & \longrightarrow & K_0^{\text{alg}}(I_{H_3}^{\text{alg}}/I_{H_1}^{\text{alg}}) & \longrightarrow & K_0^{\text{alg}}(I_{H_3}^{\text{alg}}/I_{H_2}^{\text{alg}}) \\ \gamma_{0,H_2/H_1}^E \downarrow & & \gamma_{0,H_3/H_1}^E \downarrow & & \gamma_{0,H_3/H_2}^E \downarrow \\ K_0^{\text{top}}(I_{H_2}^{\text{top}}/I_{H_1}^{\text{top}}) & \longrightarrow & K_0^{\text{top}}(I_{H_3}^{\text{top}}/I_{H_1}^{\text{top}}) & \longrightarrow & K_0^{\text{top}}(I_{H_3}^{\text{top}}/I_{H_2}^{\text{top}}) \end{array}$$

are commutative.

*Proof.* This follows from [13, Theorem 2.4.1] and [13, Theorem 3.1.9], and Remark 2.11.  $\square$

**Lemma 4.7.** *Let  $E$  be a row-finite graph. For all  $H_1, H_2$  in  $\mathcal{H}(E)$  with  $H_1 \subseteq H_2$ , we have that  $\gamma_{1,H_2/H_1}^E$  is surjective and  $\ker(\gamma_{1,H_2/H_1}^E)$  is a divisible group.*

*Proof.* We first show the lemma is true for the case  $H_2 = E^0$  and  $H_1 = \emptyset$ . Note that from Theorem 4.5, the following diagram is commutative

$$\begin{array}{ccccccc}
K_1^{\text{alg}}(\mathbb{C})^{E^0 \setminus \text{Sink}(E)} & \xrightarrow{1-B_E^t} & K_1^{\text{alg}}(\mathbb{C})^{E^0} & \longrightarrow & K_1^{\text{alg}}(L_{\mathbb{C}}(E)) & \longrightarrow & \mathbb{Z}^{E^0 \setminus \text{Sink}(E)} \xrightarrow{1-B_E^t} \mathbb{Z}^{E^0} \\
\downarrow & & \downarrow & & \downarrow \gamma_1^E & & \parallel \\
0 & \xrightarrow{1-B_E^t} & 0 & \longrightarrow & K_1^{\text{top}}(C^*(E)) & \longrightarrow & \mathbb{Z}^{E^0 \setminus \text{Sink}(E)} \xrightarrow{1-B_E^t} \mathbb{Z}^{E^0}
\end{array}$$

where the rows are exact. A diagram chase shows that  $\gamma_1^E$  is surjective and  $\text{coker}(1 - B_E^t) \cong \ker \gamma_1^E$ . Since  $K_1^{\text{alg}}(\mathbb{C})^{E^0} \cong (\mathbb{C}^\times)^{E^0}$  is a divisible group, and quotients of divisible groups are divisible, it follows that  $\text{coker}(1 - B_E^t) \cong \ker \gamma_1^E$  is a divisible group.

We now consider the case  $H_2 = H$  and  $H_1 = \emptyset$ . By Lemma 4.3, the diagram

$$\begin{array}{ccc}
L_{\mathbb{C}}(\overline{E}_{(H,\emptyset)}) & \xrightarrow{\cong} & I_H^{\text{alg}} \\
\downarrow \iota_{\overline{E}_{(H,\emptyset)}} & & \downarrow \iota_{E,H} \\
C^*(\overline{E}_{(H,\emptyset)}) & \xrightarrow{\cong} & I_H^{\text{top}}
\end{array}$$

is commutative. Therefore,

$$\begin{array}{ccc}
K_1^{\text{alg}}(L_{\mathbb{C}}(\overline{E}_{(H,\emptyset)})) & \xrightarrow{\cong} & K_1^{\text{alg}}(I_H^{\text{alg}}) \\
\downarrow \gamma_1^{\overline{E}_{(H,\emptyset)}} & & \downarrow \gamma_{1,H}^E \\
K_1^{\text{top}}(C^*(\overline{E}_{(H,\emptyset)})) & \xrightarrow{\cong} & K_1^{\text{top}}(I_H^{\text{top}})
\end{array}$$

is commutative. As in the previous case,  $\gamma_1^{\overline{E}_{(H,\emptyset)}}$  is a surjective homomorphism and  $\ker(\gamma_1^{\overline{E}_{(H,\emptyset)}})$  is a divisible group. Hence,  $\gamma_{1,H}^E$  is surjective and  $\ker(\gamma_{1,H}^E)$  is a divisible group.

Finally, suppose  $H_1$  and  $H_2$  are elements of  $\mathcal{H}(E)$  with  $H_1 \subseteq H_2$ . Then by Lemma 4.6, the diagram

$$\begin{array}{ccccccccc}
K_1^{\text{alg}}(I_{H_1}^{\text{alg}}) & \longrightarrow & K_1^{\text{alg}}(I_{H_2}^{\text{alg}}) & \longrightarrow & K_1^{\text{alg}}(I_{H_2}^{\text{alg}}/I_{H_1}^{\text{alg}}) & \longrightarrow & K_0(I_{H_1}^{\text{alg}}) & \longrightarrow & K_0^{\text{alg}}(I_{H_2}^{\text{alg}}) \\
\downarrow \gamma_{1,H_1}^E & & \downarrow \gamma_{1,H_2}^E & & \downarrow \gamma_{1,H_2/H_1}^E & & \downarrow \gamma_{0,H_1}^E & & \downarrow \gamma_{0,H_2}^E \\
K_1^{\text{top}}(I_{H_1}^{\text{top}}) & \longrightarrow & K_1^{\text{top}}(I_{H_2}^{\text{top}}) & \longrightarrow & K_1^{\text{top}}(I_{H_2}^{\text{top}}/I_{H_1}^{\text{top}}) & \longrightarrow & K_0^{\text{top}}(I_{H_1}^{\text{top}}) & \longrightarrow & K_0^{\text{top}}(I_{H_2}^{\text{top}})
\end{array}$$

is commutative and the rows are exact. By Proposition 4.4,  $\gamma_{0,H_1}^E$  and  $\gamma_{0,H_2}^E$  are isomorphism. From the previous case,  $\gamma_{1,H_2}^E$  is surjective and  $\ker(\gamma_{1,H_2}^E)$  is a divisible group. A diagram chase shows that  $\gamma_{1,H_2/H_1}^E$  is surjective and the map  $K_1^{\text{alg}}(I_{H_2}^{\text{alg}}) \rightarrow K_1^{\text{alg}}(I_{H_2}^{\text{alg}}/I_{H_1}^{\text{alg}})$  maps  $\ker(\gamma_{1,H_2}^E)$  onto  $\ker(\gamma_{1,H_2/H_1}^E)$ .

Since  $\ker(\gamma_{1,H_2}^E)$  is divisible and the quotient of a divisible group is divisible,  $\ker(\gamma_{1,H_2/H_1}^E)$  is divisible.  $\square$

**Lemma 4.8.** *Let  $G_1$  and  $G_2$  be abelian groups. Suppose  $H_i$  is a subgroup of  $G_i$  such that  $H_i$  is a divisible group and  $G_i/H_i$  is a free group for each  $i = 1, 2$ . If  $\alpha : G_1 \rightarrow G_2$  is an isomorphism, then  $\alpha(H_1) = H_2$ . Consequently, the restriction  $\alpha|_{H_1} : H_1 \rightarrow H_2$  is an isomorphism and there exists an isomorphism  $\bar{\alpha} : G_1/H_1 \rightarrow G_2/H_2$  such that the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1 & \longrightarrow & G_1 & \longrightarrow & G_1/H_1 & \longrightarrow & 0 \\ & & \downarrow \alpha|_{H_1} & & \downarrow \alpha & & \downarrow \bar{\alpha} & & \\ 0 & \longrightarrow & H_2 & \longrightarrow & G_2 & \longrightarrow & G_2/H_2 & \longrightarrow & 0 \end{array}$$

is commutative.

*Proof.* Let  $\overline{\alpha(H_1)}$  be the image of  $\alpha(H_1)$  in  $G_2/H_2$ . Since  $H_1$  is a divisible group,  $\overline{\alpha(H_1)}$  is a divisible group. Since  $G_2/H_2$  is a free group, we have that  $\overline{\alpha(H_1)} = 0$ . Thus,  $\alpha(H_1) \subseteq H_2$ . Similar,  $\alpha^{-1}(H_2) \subseteq H_1$ . The lemma now follows.  $\square$

**Theorem 4.9.** *Let  $E_1$  and  $E_2$  be row-finite graphs satisfying Condition (K). If  $K_{\text{ideal}}^{\text{alg},+}(L_{\mathbb{C}}(E_1))$  is isomorphic to  $K_{\text{ideal}}^{\text{alg},+}(L_{\mathbb{C}}(E_2))$ , then  $K_{\text{ideal}}^{\text{top},+}(C^*(E_1))$  is isomorphic to  $K_{\text{ideal}}^{\text{top},+}(C^*(E_2))$ . Moreover, the isomorphism can be chosen such that the diagram*

$$\begin{array}{ccc} K_m^{\text{alg}}(L_{\mathbb{C}}(E_1)) & \xrightarrow{\cong} & K_m^{\text{alg}}(L_{\mathbb{C}}(E_2)) \\ \gamma_m^{E_1} \downarrow & & \downarrow \gamma_m^{E_2} \\ K_m^{\text{top}}(C^*(E_1)) & \xrightarrow{\cong} & K_m^{\text{top}}(C^*(E_2)) \end{array}$$

is commutative for  $m = 0, 1$ .

*Proof.* Suppose  $K_{\text{ideal}}^{\text{alg},+}(L_{\mathbb{C}}(E_1)) \cong K_{\text{ideal}}^{\text{alg},+}(L_{\mathbb{C}}(E_2))$ . By Remark 3.12, there exists a lattice isomorphism  $\beta : \mathcal{H}(E_1) \rightarrow \mathcal{H}(E_2)$  and for each pair  $H_1$  and  $H_2$  in  $\mathcal{H}(E_1)$  with  $H_1 \subseteq H_2$ , there exist group isomorphisms

$$\lambda_n^{H_1, H_2} : K_n^{\text{alg}}(I_{H_2}^{\text{alg}}/I_{H_1}^{\text{alg}}) \rightarrow K_n^{\text{alg}}(I_{\beta(H_2)}^{\text{alg}}/I_{\beta(H_1)}^{\text{alg}}) \text{ for all } n \in \mathbb{Z}$$

with  $\lambda_0^{H_1, H_2}$  an order isomorphism, and this collection preserves all natural transformations.

Let  $H_1, H_2 \in \mathcal{H}(E_1)$  with  $H_1 \subseteq H_2$ . Since  $\lambda_m^{H_1, H_2}$  is an isomorphism for  $m = 0, 1$ , by Proposition 4.4, Lemma 4.8, and Lemma 4.7, there exists an isomorphism

$$\alpha_m^{H_1, H_2} : K_m^{\text{top}}(I_{H_1}^{\text{top}}/I_{H_2}^{\text{top}}) \rightarrow K_m^{\text{top}}(I_{\beta(H_2)}^{\text{top}}/I_{\beta(H_1)}^{\text{top}})$$

such that the diagram

$$\begin{array}{ccc}
K_m^{\text{alg}} \left( I_{H_2}^{\text{alg}} / I_{H_1}^{\text{alg}} \right) & \xrightarrow{\lambda_m^{H_1, H_2}} & K_m^{\text{alg}} \left( I_{\beta(H_2)}^{\text{alg}} / I_{\beta(H_1)}^{\text{alg}} \right) \\
\gamma_{m, H_2 / H_1}^{E_1} \downarrow & & \downarrow \gamma_{m, \beta(H_2) / \beta(H_1)}^{E_2} \\
K_m^{\text{top}} \left( I_{H_2}^{\text{top}} / I_{H_1}^{\text{top}} \right) & \xrightarrow{\alpha_m^{H_1, H_2}} & K_m^{\text{top}} \left( I_{\beta(H_2)}^{\text{top}} / I_{\beta(H_1)}^{\text{top}} \right)
\end{array}$$

is commutative for  $m = 0, 1$  and  $\alpha_0^{H_1, H_1}$  is an order isomorphism.

Since the collection

$$\phi := \left( \{ \lambda_n^{H_1, H_2} : H_1, H_2 \in \mathcal{H}(E_1) \text{ with } H_1 \subseteq H_2, n \in \mathbb{Z} \}, \beta \right)$$

is an isomorphism from  $K_{\text{ideal}}^{\text{alg}, +}(L_K(E_1))$  to  $K_{\text{ideal}}^{\text{alg}, +}(L_K(E_2))$ , by Lemma 4.6 and Remark 3.12, the collection

$$\psi := \left( \{ \alpha_n^{H_1, H_2} : H_1, H_2 \in \mathcal{H}(E_1) \text{ with } H_1 \subseteq H_2, n \in \mathbb{Z} \}, \beta \right)$$

is an isomorphism from  $K_{\text{ideal}}^{\text{top}, +}(C^*(E_1))$  to  $K_{\text{ideal}}^{\text{top}, +}(C^*(E_2))$ .  $\square$

**Theorem 4.10.** *Let  $\mathcal{C}$  be a class of graphs that satisfies the following two properties:*

- (1) *Every graph in  $\mathcal{C}$  satisfies Condition (K).*
- (2) *If  $E, F \in \mathcal{C}$  and  $K_{\text{ideal}}^{\text{top}, +}(C^*(E)) \cong K_{\text{ideal}}^{\text{top}, +}(C^*(F))$ , then  $C^*(E)$  is strongly Morita equivalent to  $C^*(F)$ .*

*Then the Morita equivalence conjecture holds for all graphs in  $\mathcal{C}$ . In other words, if  $E, F \in \mathcal{C}$  and  $L_{\mathbb{C}}(E)$  is Morita equivalent to  $L_{\mathbb{C}}(F)$ , then  $C^*(E)$  is strongly Morita equivalent to  $C^*(F)$ .*

*Proof.* Suppose  $E, F \in \mathcal{C}$  and  $L_{\mathbb{C}}(E)$  is Morita equivalent to  $L_{\mathbb{C}}(F)$ . By [15, Theorem 2.11] and [2, Theorem 5.2] there exists a row-finite graph  $\tilde{E}$  satisfying Condition (K) such that  $L_{\mathbb{C}}(\tilde{E})$  is Morita equivalent to  $L_{\mathbb{C}}(E)$  and  $C^*(\tilde{E})$  is strongly Morita equivalent to  $C^*(E)$ . (Such a graph  $\tilde{E}$  is called a *desingularization* of  $E$ .) Likewise, there exists a row-finite graph  $\tilde{F}$  satisfying Condition (K) such that  $L_{\mathbb{C}}(\tilde{F})$  is Morita equivalent to  $L_{\mathbb{C}}(F)$  and  $C^*(\tilde{F})$  is strongly Morita equivalent to  $C^*(F)$ .

Let  $S\tilde{E}$  be the stabilization graph of  $\tilde{E}$ , formed by adding an infinite head to each vertex of  $\tilde{E}$ . It follows from [3, Proposition 9.8] that  $L_{\mathbb{C}}(S\tilde{E}) \cong M_{\infty}(L_{\mathbb{C}}(\tilde{E}))$  (as  $*$ -algebras) and  $C^*(S\tilde{E}) \cong C^*(\tilde{E}) \otimes \mathbb{K}$  (as  $*$ -algebras). Likewise, if  $S\tilde{F}$  is the stabilization graph of  $\tilde{F}$ , then  $L_{\mathbb{C}}(S\tilde{F}) \cong M_{\infty}(L_{\mathbb{C}}(\tilde{F}))$  (as  $*$ -algebras) and  $C^*(S\tilde{F}) \cong C^*(\tilde{F}) \otimes \mathbb{K}$  (as  $*$ -algebras).

Since  $L_{\mathbb{C}}(E)$  is Morita equivalent to  $L_{\mathbb{C}}(F)$ , it follows that  $L_{\mathbb{C}}(\tilde{E})$  is Morita equivalent to  $L_{\mathbb{C}}(\tilde{F})$ . By [3, Proposition 9.10]  $M_{\infty}(L_{\mathbb{C}}(\tilde{E})) \cong M_{\infty}(L_{\mathbb{C}}(\tilde{F}))$  (as rings). Thus  $L_{\mathbb{C}}(S\tilde{E}) \cong L_{\mathbb{C}}(S\tilde{F})$  (as rings). Hence  $K_{\text{ideal}}^{\text{alg}, +}(L_{\mathbb{C}}(S\tilde{E})) \cong K_{\text{ideal}}^{\text{alg}, +}(L_{\mathbb{C}}(S\tilde{F}))$ . Since  $\tilde{E}$  and  $\tilde{F}$  are row-finite and satisfy Condition (K),



and since the process of stabilizing a graph preserves row-finiteness and Condition (K), Theorem 4.9 implies  $K_{\text{ideal}}^{\text{top},+}(C^*(S\tilde{E})) \cong K_{\text{ideal}}^{\text{top},+}(C^*(S\tilde{F}))$ .

In addition, since  $C^*(E)$  is strongly Morita equivalent to  $C^*(S\tilde{E})$ , and  $C^*(F)$  is strongly Morita equivalent to  $C^*(S\tilde{F})$ , it follows that

$$K_{\text{ideal}}^{\text{top},+}(C^*(E)) \cong K_{\text{ideal}}^{\text{top},+}(C^*(S\tilde{E})) \quad \text{and} \quad K_{\text{ideal}}^{\text{top},+}(C^*(F)) \cong K_{\text{ideal}}^{\text{top},+}(C^*(S\tilde{F})).$$

Hence

$$K_{\text{ideal}}^{\text{top},+}(C^*(E)) \cong K_{\text{ideal}}^{\text{top},+}(C^*(F)).$$

By hypothesis, we then have  $C^*(E)$  is strongly Morita equivalent to  $C^*(F)$ .  $\square$

**Remark 4.11.** Define

$$\mathcal{C} := \{E : E \text{ is a graph and } C^*(E) \text{ has finitely many ideals}\}$$

and note that  $\mathcal{C}$  coincides with the class of graphs  $E$  such that  $L_{\mathbb{C}}(E)$  has finitely many ideals. It follows from basic graph algebra results that every graph in  $\mathcal{C}$  satisfies Condition (K), and hence  $\mathcal{C}$  satisfies Property (1) of Theorem 4.10. It was boldly conjectured in [20] that the  $C^*$ -algebras of the graphs in  $\mathcal{C}$  are determined up to stable isomorphism by their ideal-related topological  $K$ -theory; that is, it was conjectured that  $\mathcal{C}$  satisfies Property (2) of Theorem 4.10. If this conjecture is true then Theorem 4.10 implies that the Morita equivalence conjecture of [3, §9] holds for all graphs in  $\mathcal{C}$ .

Although it is not known if all graphs with  $C^*$ -algebras having finitely many ideals are determined up to stable isomorphism by their ideal-related topological  $K$ -theory, special cases of this conjecture have been established for many subclasses of  $\mathcal{C}$ . This allows us to establish a number of corollaries using these results.

**Corollary 4.12.** *If  $E$  and  $F$  are graphs such that  $C^*(E)$  and  $C^*(F)$  have exactly one proper, nonzero ideal, and if  $L_{\mathbb{C}}(E)$  is Morita equivalent to  $L_{\mathbb{C}}(F)$ , then  $C^*(E)$  is strongly Morita equivalent to  $C^*(F)$ .*

*Proof.* Let

$$\mathcal{C} := \{E : E \text{ is a graph and } C^*(E) \text{ has exactly one proper, nonzero ideal}\}.$$

Since every graph whose associated  $C^*$ -algebra has a finite number of ideals must satisfy Condition (K),  $\mathcal{C}$  satisfies Property (1) of Theorem 4.10. It follows from [22, Theorem 4.5] that  $\mathcal{C}$  satisfies Property (2) of Theorem 4.10.  $\square$

**Corollary 4.13.** *If  $E$  and  $F$  are graphs such that each of  $C^*(E)$  and  $C^*(F)$  have a largest proper ideal that is also AF, and if  $L_{\mathbb{C}}(E)$  is Morita equivalent to  $L_{\mathbb{C}}(F)$ , then  $C^*(E)$  is strongly Morita equivalent to  $C^*(F)$ .*

*Proof.* Let

$$\mathcal{C} := \{E : E \text{ is a graph, } C^*(E) \text{ has a largest ideal } I, \text{ and } I \text{ is AF}\}.$$

It follows from basic graph algebra results that a graph whose  $C^*$ -algebra has a largest ideal that is also AF must satisfy Condition (K). Thus  $\mathcal{C}$  satisfies Property (1) of Theorem 4.10. It follows from [22, Theorem 4.6] that  $\mathcal{C}$  satisfies Property (2) of Theorem 4.10.  $\square$

**Corollary 4.14.** *If  $E$  and  $F$  are graphs such that each of  $C^*(E)$  and  $C^*(F)$  have a smallest nonzero ideal that is purely infinite and whose corresponding quotient is AF, and if  $L_{\mathbb{C}}(E)$  is Morita equivalent to  $L_{\mathbb{C}}(F)$ , then  $C^*(E)$  is strongly Morita equivalent to  $C^*(F)$ .*

*Proof.* Let

$$\mathcal{C} := \{E : E \text{ is a graph, } C^*(E) \text{ has a smallest ideal } I, \text{ the ideal } I \text{ is purely infinite, and } C^*(E)/I \text{ is AF}\}.$$

It follows from basic graph algebra results that any graph in  $\mathcal{C}$  must satisfy Condition (K). Thus  $\mathcal{C}$  satisfies Property (1) of Theorem 4.10. It follows from [16, Corollary 6.4] that  $\mathcal{C}$  satisfies Property (2) of Theorem 4.10.  $\square$

**Corollary 4.15.** *If  $E$  and  $F$  are finite graphs with no sinks satisfying Condition (K), and if  $L_{\mathbb{C}}(E)$  is Morita equivalent to  $L_{\mathbb{C}}(F)$ , then  $C^*(E)$  is strongly Morita equivalent to  $C^*(F)$ .*

*Proof.* Let

$$\mathcal{C} := \{E : E \text{ is a finite graph with no sinks satisfying Condition (K)}\}.$$

By definition, we have that  $\mathcal{C}$  satisfies Property (1) of Theorem 4.10. If  $E$  is a finite graph satisfying Condition (K), then by the process of source removal (see [27, Proposition 3.1]) there exists a finite graph  $F$  with no sinks or sources such that satisfies Condition (K), and has the properties that  $L_{\mathbb{C}}(E)$  is Morita equivalent to  $L_{\mathbb{C}}(F)$  and  $C^*(E)$  is strongly Morita equivalent to  $C^*(F)$ . Since  $F$  is a finite graph with no sinks or sources satisfying Condition (K),  $C^*(F)$  is (isomorphic to) a Cuntz-Krieger algebra of a matrix satisfying Condition (II). Restorff has shown in [24] that all Cuntz-Krieger algebras associated with matrices satisfying Condition (II) are classified up to stable isomorphism by their ideal-related topological  $K$ -theory. Hence  $\mathcal{C}$  satisfies Property (2) of Theorem 4.10.  $\square$

**Corollary 4.16.** *If  $E$  and  $F$  are amplified graphs (i.e., between any two vertices there are either no edges or infinitely many edges) with finitely many vertices, and if  $L_{\mathbb{C}}(E)$  is Morita equivalent to  $L_{\mathbb{C}}(F)$ , then  $C^*(E)$  is strongly Morita equivalent to  $C^*(F)$ .*

*Proof.* Let

$$\mathcal{C} := \{E : E \text{ is an amplified graph with a finite number of vertices}\}.$$

Every amplified graph satisfies Condition (K), and thus  $\mathcal{C}$  satisfies Property (1) of Theorem 4.10. It follows from [21, Theorem 5.7] that  $\mathcal{C}$  satisfies Property (2) of Theorem 4.10.  $\square$

**Corollary 4.17.** *If  $E$  and  $F$  are graphs whose  $C^*$ -algebras are purely infinite and have primitive ideal spaces that are accordion spaces in the sense of [10, Definition 1.1], and if  $L_{\mathbb{C}}(E)$  is Morita equivalent to  $L_{\mathbb{C}}(F)$ , then  $C^*(E)$  is strongly Morita equivalent to  $C^*(F)$ .*

*Proof.* Let

$$\mathcal{C} := \{E : E \text{ is a graph, } C^*(E) \text{ is purely infinite, and } \text{Prim } C^*(E) \text{ is an accordion space}\}.$$

Any graph  $C^*$ -algebra with a primitive ideal space that is an accordion space has a finite number of ideals, and hence must come from a graph satisfying Condition (K). Thus  $\mathcal{C}$  satisfies Property (1) of Theorem 4.10. It follows from [9] and [10, Theorem 1.3] that  $\mathcal{C}$  satisfies Property (2) of Theorem 4.10.  $\square$

The following result extends [3, Corollary 9.16] to non-row-finite graphs.

**Corollary 4.18.** *If  $E$  and  $F$  are graphs whose  $C^*$ -algebras are simple, and if  $L_{\mathbb{C}}(E)$  is Morita equivalent to  $L_{\mathbb{C}}(F)$ , then  $C^*(E)$  is strongly Morita equivalent to  $C^*(F)$ .*

*Proof.* Let

$$\mathcal{C} := \{E : E \text{ is a graph and } C^*(E) \text{ is simple}\}.$$

Since any simple graph  $C^*$ -algebra must come from a graph that satisfies Condition (K), and thus  $\mathcal{C}$  satisfies Property (1) of Theorem 4.10. It follows from the dichotomy of simple graph  $C^*$ -algebras that any simple graph  $C^*$ -algebra is either AF or purely infinite. Hence Elliott's theorem and the Kirchberg-Phillips classification theorem imply that simple graph  $C^*$ -algebras are determined up to stable isomorphism by their ordered  $K$ -theory. Since ideal-related topological  $K$ -theory reduces to ordered  $K$ -theory in the simple case,  $\mathcal{C}$  satisfies Property (2) of Theorem 4.10.  $\square$

## 5. THE ISOMORPHISM CONJECTURE FOR GRAPH ALGEBRAS

In this section we consider the isomorphism conjecture for graph algebras. As in the previous section, we consider ideal-related  $K$ -theory, however, now we must also keep track of the position of the class of the unit in the  $K_0$ -group. We prove that if the ideal-related algebraic  $K$ -theories of two unital Leavitt path algebras of graphs satisfying Condition (K) are isomorphic via an isomorphism taking the class of the unit to the class of the unit, then there is an isomorphism between the ideal-related topological  $K$ -theories of the graph  $C^*$ -algebras taking the class of the unit to the class of the unit. This allows us to prove that the isomorphism conjecture for graph algebras holds for any class of graphs whose  $C^*$ -algebras are classified up to isomorphism by their ideal-related topological  $K$ -theory plus the position of the class of the unit. By applying existing classification theorems, we confirm the isomorphism conjecture for a number of specific classes of graphs in the corollaries at the end of this section.

**Lemma 5.1.** *Let  $R$  be a ring and let  $e \in R$  be an idempotent such that  $ReR = R$ . Let  $I$  be an ideal of  $R$ . If  $I$  has a countable approximate unit consisting of idempotents, then the ideal of  $R$  generated by  $eIe$  is equal to  $I$ . Moreover, if every ideal of  $R$  has a countable approximate unit consisting of idempotents, then*

$$I \mapsto eIe$$

*is a lattice isomorphism from  $\text{Lat}^{\text{alg}}(R)$  to  $\text{Lat}^{\text{alg}}(eRe)$  with inverse given by*

$$I \mapsto \text{the ideal in } R \text{ generated by } I.$$

*Proof.* Suppose  $\{u_n\}_{n=1}^\infty$  is a countable approximate unit consisting of idempotents for  $I$ . We will first show that  $IeI = I$ , where  $IeI$  is the ideal of  $I$  given by

$$IeI := \left\{ er + se + \sum_{i=1}^m r_i e s_i : r, s, r_i, s_i \in I, m \in \mathbb{N} \right\}.$$

It is clear that  $IeI \subseteq I$ .

Let  $x \in I$ . Since  $ReR = R$ , we have that  $x \in ReR$ . Since  $u_n \in I$  for each  $n \in \mathbb{N}$ , we have that  $u_n x u_n \in IeI$  for each  $n \in \mathbb{N}$ . Since  $\{u_n\}_{n=1}^\infty$  is an approximate unit of  $I$ , there exists  $n \in \mathbb{N}$  such that  $u_n x u_n = x$ . Therefore,  $x = u_n x u_n \in IeI$ . Thus  $I \subseteq IeI$ , and  $IeI = I$ .

Let  $J$  be the ideal of  $R$  generated by  $eIe$ . Since  $eIe \subseteq I$  and  $I$  is an ideal of  $R$ , we have that  $J \subseteq I$ . Let  $x \in I$ . Since  $IeI = I$ , there exist  $m \in \mathbb{N}$ ,  $r, s, r_i, s_i \in I$  for  $i = 1, 2, \dots, m$  such that

$$x = er + se + \sum_{i=1}^m r_i e s_i.$$

Since  $er \in I$ , there exists  $n_1 \in \mathbb{N}$  such that  $u_{n_1} er = er$ . Hence

$$er = eer = eu_{n_1} er \in (eIe)R \subseteq J.$$

Likewise, since  $se \in I$ , there exists  $n_2 \in \mathbb{N}$  such that  $seu_{n_2} = se$ . Hence,

$$se = see = seu_{n_2} e \in R(eIe) \subseteq J.$$

Finally, for each  $1 \leq i \leq m$ , we have  $es_i \in I$  and hence there exists  $k_i \in \mathbb{N}$  such that  $u_{k_i} es_i = es_i$ . Hence

$$r_i e s_i = r_i e e s_i = r_i e u_{k_i} es_i \in R(eIe)R \subseteq J.$$

Thus we have

$$x = er + se + \sum_{i=1}^m r_i e s_i \in J$$

and we may conclude that  $I \subseteq J$  and  $I = J$ .

Moreover, if every ideal of  $R$  has a countable approximate unit consisting of idempotents, then it is straightforward to see that the map  $\mu : \text{Lat}^{\text{alg}}(R) \rightarrow \text{Lat}^{\text{alg}}(eRe)$  defined by  $\mu(I) = eIe$  has inverse

$$\rho : \text{Lat}^{\text{alg}}(eRe) \rightarrow \text{Lat}^{\text{alg}}(R)$$

defined by  $\rho(J)$  equals the ideal in  $R$  generated by  $J$ ; indeed, the fact that  $\rho \circ \mu = \text{id}$  is the result of the first part of this proof, and the fact that  $\mu \circ \rho = \text{id}$  is straightforward to verify.

In addition,  $\mu$  preserves inclusion, and since any bijection between lattices that preserves inclusion is a lattice isomorphism, we may conclude that  $\mu$  is a lattice isomorphism.  $\square$

**Lemma 5.2.** *Let  $R$  be a ring containing a countable approximate unit consisting of idempotents. If  $e$  is an idempotent of  $R$  such that  $ReR = R$ , then the homomorphism  $K_n^{\text{alg}}(\iota) : K_n^{\text{alg}}(eRe) \rightarrow K_n^{\text{alg}}(R)$  induced by the inclusion  $\iota : eRe \rightarrow R$  is an isomorphism for all  $n \in \mathbb{Z}$ .*

*Proof.* If  $R$  is unital, this is a well-known fact (see [4, Lemma 2.7]). If  $R$  is not unital, we shall show that we can obtain the result from the unital case by a direct limit argument.

Let  $\{e_k\}_{k=1}^\infty$  be a countable approximate unit for  $R$  consisting of idempotents. Without loss of generality, we may assume that  $e_k e = e e_k = e$  for all  $k \in \mathbb{N}$ . Also,  $R = \bigcup_{k=1}^\infty e_k R e_k$  (see Remark 3.7).

Since  $eRe = e_k(eRe)e_k$ , we will define  $\phi_k : eRe \rightarrow e_k R e_k$  to be the inclusion map with  $\phi_k(x) := x$ . We see that  $eRe$  is a full corner of  $e_k R e_k$  because

$$(e_k R e_k)e(e_k R e_k) = e_k R(e_k e e_k) R e_k = e_k (R e R) e_k = e_k (R) e_k.$$

Thus, by [4, Lemma 2.7] for every  $n \in \mathbb{Z}$ ,

$$K_n^{\text{alg}}(\phi_k) : K_n^{\text{alg}}(eRe) \rightarrow K_n^{\text{alg}}(e_k R e_k)$$

is an isomorphism. If  $\iota_{k,k+1} : e_k R e_k \rightarrow e_{k+1} R e_{k+1}$  is the inclusion map, we see that  $\iota_{k,k+1} \circ \phi_k = \phi_{k+1}$ . Hence

$$K_n^{\text{alg}}(\iota_{k,k+1}) \circ K_n^{\text{alg}}(\phi_k) = K_n^{\text{alg}}(\phi_{k+1}).$$

By the universal property of the direct limit the induced map from  $K_n^{\text{alg}}(eRe)$  to  $\varinjlim (K_n^{\text{alg}}(e_k R e_k), K_n^{\text{alg}}(\iota_{k,k+1})) = K_n^{\text{alg}}(R)$  is an isomorphism. But this induced map is precisely  $K_n^{\text{alg}}(\iota)$ .  $\square$

**Lemma 5.3.** *Let  $R$  and  $S$  be rings such that every subquotient of  $R$  satisfies excision in algebraic  $K$ -theory and every subquotient of  $S$  satisfies excision in algebraic  $K$ -theory, and let  $\phi : R \rightarrow S$  be a homomorphism. Suppose the following two conditions are satisfied:*

- (1) *the map  $\beta : \text{Lat}^{\text{alg}}(R) \rightarrow \text{Lat}^{\text{alg}}(S)$  defined by*

$$\beta(I) := \text{the ideal in } S \text{ generated by } \phi(I)$$

*is a lattice isomorphism, and*

- (2) *for each  $I_1, I_2 \in \text{Lat}^{\text{alg}}(R)$  with  $I_1 \subseteq I_2$ , the homomorphism  $\phi_{I_2/I_1} : I_2/I_1 \rightarrow \beta(I_2)/\beta(I_1)$  defined by*

$$\phi_{I_2/I_1}(x + I_1) := \phi(x) + \beta(I_1)$$

induces an isomorphism

$$K_n^{\text{alg}}(\phi_{I_2/I_1}) : K_n^{\text{alg}}(I_2/I_1) \rightarrow K_n^{\text{alg}}(\beta(I_2)/\beta(I_1))$$

for each  $n \in \mathbb{Z}$  with  $K_0^{\text{alg}}(\phi_{I_2/I_1})$  also being an order isomorphism.

Then the collection

$$\left\{ K_n^{\text{alg}}(\phi_{I_2/I_1}) : I_1, I_2 \in \text{Lat}^{\text{alg}}(R), I_1 \subseteq I_2, \text{ and } n \in \mathbb{Z} \right\},$$

together with the lattice isomorphism  $\beta$ , is an isomorphism of ideal-related algebraic  $K$ -theory, and  $K_{\text{ideal}}^{\text{alg},+}(R) \cong K_{\text{ideal}}^{\text{alg},+}(S)$ .

*Proof.* Note that for  $I_1, I_2, I_3 \in \text{Lat}^{\text{alg}}(R)$  with  $I_1 \subseteq I_2 \subseteq I_3$ , we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_2/I_1 & \longrightarrow & I_3/I_1 & \longrightarrow & I_3/I_2 & \longrightarrow & 0 \\ & & \downarrow \phi_{I_2/I_1} & & \downarrow \phi_{I_3/I_1} & & \downarrow \phi_{I_3/I_2} & & \\ 0 & \longrightarrow & \beta(I_2)/\beta(I_1) & \longrightarrow & \beta(I_3)/\beta(I_1) & \longrightarrow & \beta(I_3)/\beta(I_2) & \longrightarrow & 0 \end{array}$$

Applying the functor  $K_n^{\text{alg}}$  we obtain the following commutative diagram:

$$\begin{array}{ccccccc} K_n^{\text{alg}}(I_2/I_1) & \longrightarrow & K_n^{\text{alg}}(I_3/I_1) & \longrightarrow & K_n^{\text{alg}}(I_3/I_2) & \longrightarrow & K_{n-1}^{\text{alg}}(I_2/I_1) \\ \downarrow K_n^{\text{alg}}(\phi_{I_2/I_1}) & & \downarrow K_n^{\text{alg}}(\phi_{I_3/I_1}) & & \downarrow K_n^{\text{alg}}(\phi_{I_3/I_2}) & & \downarrow K_{n-1}^{\text{alg}}(\phi_{I_2/I_1}) \\ K_n^{\text{alg}}(\beta(I_2)/\beta(I_1)) & \longrightarrow & K_n^{\text{alg}}(\beta(I_3)/\beta(I_1)) & \longrightarrow & K_n^{\text{alg}}(\beta(I_3)/\beta(I_2)) & \longrightarrow & K_{n-1}^{\text{alg}}(\beta(I_2)/\beta(I_1)) \end{array}$$

Thus  $\beta$  and the collection of  $K_n^{\text{alg}}(\phi_{I_2/I_1})$  for all  $n \in \mathbb{Z}$  and  $I_1, I_2 \in \text{Lat}^{\text{alg}}(R)$  with  $I_1 \subseteq I_2$  give an isomorphism of ideal-related algebraic  $K$ -theories, and  $K_{\text{ideal}}^{\text{alg},+}(R) \cong K_{\text{ideal}}^{\text{alg},+}(S)$ .  $\square$

We have a similar result for  $C^*$ -algebras.

**Lemma 5.4.** *Let  $\mathfrak{A}$  and  $\mathfrak{B}$  be  $C^*$ -algebras, and let  $\phi : \mathfrak{A} \rightarrow \mathfrak{B}$  be a homomorphism. Suppose*

- (1) *the map  $\beta : \text{Lat}^{\text{top}}(\mathfrak{A}) \rightarrow \text{Lat}^{\text{top}}(\mathfrak{B})$  defined by*

$$\beta(\mathfrak{I}) := \text{the ideal in } \mathfrak{B} \text{ generated by } \phi(\mathfrak{I})$$

*is a lattice isomorphism; and*

- (2) *for each  $\mathfrak{I}_1, \mathfrak{I}_2 \in \text{Lat}^{\text{alg}}(\mathfrak{A})$  with  $\mathfrak{I}_1 \subseteq \mathfrak{I}_2$  the homomorphism  $\phi_{\mathfrak{I}_2/\mathfrak{I}_1} : \mathfrak{I}_2/\mathfrak{I}_1 \rightarrow \beta(\mathfrak{I}_2)/\beta(\mathfrak{I}_1)$  defined by*

$$\phi_{\mathfrak{I}_2/\mathfrak{I}_1}(x + \mathfrak{I}_1) := \phi(x) + \beta(\mathfrak{I}_1)$$

*induces an isomorphism*

$$K_n^{\text{top}}(\phi_{\mathfrak{I}_2/\mathfrak{I}_1}) : K_n^{\text{top}}(\mathfrak{I}_2/\mathfrak{I}_1) \rightarrow K_n^{\text{top}}(\beta(\mathfrak{I}_2)/\beta(\mathfrak{I}_1))$$

*for each  $n = 0, 1$  with  $K_0^{\text{top}}(\phi_{\mathfrak{I}_2/\mathfrak{I}_1})$  also being an order isomorphism.*

Then the collection

$$\{K_n^{\text{top}}(\phi_{\mathfrak{I}_2/\mathfrak{I}_1}) : \mathfrak{I}_1, \mathfrak{I}_2 \in \text{Lat}^{\text{top}}(\mathfrak{A}), \mathfrak{I}_1 \subseteq \mathfrak{I}_2, \text{ and } n = 0, 1\}$$

with the lattice isomorphism  $\beta$  gives an isomorphism of ideal-related topological  $K$ -theories, and  $K_{\text{ideal}}^{\text{top},+}(\mathfrak{A}) \cong K_{\text{ideal}}^{\text{top},+}(\mathfrak{B})$ .

*Proof.* Note that every  $C^*$ -algebra satisfies excision in topological  $K$ -theory. Using the same argument as in the proof of Lemma 5.3 we get the desired result.  $\square$

**Definition 5.5.** We establish notation and terminology for the situations in Lemma 5.3 and Lemma 5.4.

Let  $R$  and  $S$  be rings such that every subquotient of  $R$  satisfies excision in algebraic  $K$ -theory and every subquotient of  $R$  satisfies excision in algebraic  $K$ -theory. If  $\phi : R \rightarrow S$  is a homomorphism satisfying the assumptions of Lemma 5.3, we write

$$[\phi] := \left( \left\{ K_n^{\text{alg}}(\phi_{I_2/I_1}) : I_1, I_2 \in \text{Lat}^{\text{alg}}(R) \text{ with } I_1 \subseteq I_2, n \in \mathbb{Z} \right\}, \beta \right)$$

and say that  $\phi$  induces an isomorphism  $[\phi] : K_{\text{ideal}}^{\text{alg},+}(R) \rightarrow K_{\text{ideal}}^{\text{alg},+}(S)$ .

Let  $\mathfrak{A}$  and  $\mathfrak{B}$  be  $C^*$ -algebras. If  $\phi : \mathfrak{A} \rightarrow \mathfrak{B}$  is a homomorphism satisfying the assumptions in Lemma 5.4, then we write

$$[\phi] := \left( \left\{ K_n^{\text{top}}(\phi_{\mathfrak{I}_2/\mathfrak{I}_1}) : \mathfrak{I}_1, \mathfrak{I}_2 \in \text{Lat}^{\text{top}}(\mathfrak{A}) \text{ with } \mathfrak{I}_1 \subseteq \mathfrak{I}_2, n = 0, 1 \right\}, \beta \right)$$

and say that  $\phi$  induces an isomorphism  $[\phi] : K_{\text{ideal}}^{\text{top},+}(\mathfrak{A}) \rightarrow K_{\text{ideal}}^{\text{top},+}(\mathfrak{B})$ .

**Lemma 5.6.** *Let  $R$  be a ring with a countable approximate unit consisting of idempotents. If  $x \in K_0^{\text{alg}}(R)$ , then there exist idempotents  $e \in M_k(R)$  and  $f \in M_k(R)$  such that  $x = [e] - [f]$ .*

*Proof.* If  $R$  is unital, then this is clear by the definition of  $K_0^{\text{alg}}(R)$ . Suppose  $R$  is a ring with a countable approximate unit consisting of idempotents  $\{e_n\}_{n=1}^\infty$ . Let  $\iota_{n,n+1} : e_n R e_n \rightarrow e_{n+1} R e_{n+1}$  and  $\iota_n : e_n R e_n \rightarrow R$  be the inclusion homomorphisms. Then  $R \cong \varinjlim (e_n R e_n, \iota_{n,n+1})$ , and by continuity of  $K$ -theory

$$K_0^{\text{alg}}(R) \cong \varinjlim (K_0^{\text{alg}}(e_n R e_n), K_0^{\text{alg}}(\iota_{n,n+1})).$$

Let  $\alpha : \varinjlim (K_0^{\text{alg}}(e_n R e_n), K_0^{\text{alg}}(\iota_{n,n+1})) \rightarrow K_0^{\text{alg}}(R)$  denote this isomorphism and let  $\psi_n : K_0^{\text{alg}}(e_n R e_n) \rightarrow \varinjlim (K_0^{\text{alg}}(e_n R e_n), K_0^{\text{alg}}(\iota_{n,n+1}))$  be the natural map to the inductive limit. Then for each  $n \in \mathbb{N}$  we have  $\alpha \circ \psi_n = K_0^{\text{alg}}(\iota_n)$ .

Let  $x \in K_0^{\text{alg}}(R)$ . Then there exists  $y \in \varinjlim (K_0^{\text{alg}}(e_n R e_n), K_0^{\text{alg}}(\iota_{n,n+1}))$  such that  $\alpha(y) = x$ . Hence, there exist  $n \in \mathbb{N}$  and  $y_n \in K_0^{\text{alg}}(e_n R e_n)$  such that  $\psi_n(y_n) = y$ . Since  $e_n R e_n$  is a unital ring,  $y_n = [a_n] - [b_n]$  for some idempotents  $a_n, b_n \in M_k(e_n R e_n)$ . Hence, if we let  $e = \iota_n(a_n)$  and  $f = \iota_n(b_n)$ , then

$$x = \alpha(y) = (\alpha \circ \psi_n)(y_n) = K_0^{\text{alg}}(\iota_n)([a_n] - [b_n]) = [e] - [f].$$

□

**Lemma 5.7.** *Let  $R$  be a ring and let  $e$  be an idempotent such that  $ReR = R$ . Let  $I_1, I_2 \in \text{Lat}^{\text{alg}}(R)$  with  $I_1 \subseteq I_2$ . Then for every idempotent  $p \in M_n(I_2/I_1)$ , there exists an idempotent  $q \in M_k(eI_2e/eI_1e)$  such that  $[p] = [q]$  in  $K_0^{\text{alg}}(I_2/I_1)$ .*

*Proof.* Let  $I_1, I_2 \in \text{Lat}^{\text{alg}}(R)$  such that  $I_1 \subseteq I_2$  and let  $p$  be an idempotent in  $M_n(I_2/I_1)$ . Since  $ReR = R$ , we have  $(R/I_1)\bar{e}(R/I_1) = R/I_1$ , where  $\bar{e} = e + I \in R/I$ . For each  $n \in \mathbb{N}$ , define  $\bar{e}_n = \text{diag}(\bar{e}, \dots, \bar{e}) \in M_n(R/I)$ .

Since  $(R/I_1)\bar{e}(R/I_1) = R/I_1$ , we have  $M_n(R/I_1)\bar{e}_n M_n(R/I_1) = M_n(R/I_1)$  for all  $n \in \mathbb{N}$ . Hence,  $p \in M_n(R/I_1)\bar{e}_n M_n(R/I_1)$  and there exist

$$x_1, \dots, x_m, y_1, \dots, y_m \in M_n(R/I_1)$$

and  $\ell \in \mathbb{Z}$  with  $\ell \geq 0$  such that

$$p = \ell \bar{e}_n + x_1 \bar{e}_n + \bar{e}_n y_1 + \sum_{i=2}^m x_i \bar{e}_n y_i.$$

Since  $p$  is an idempotent,

$$p = \ell p \bar{e}_n p + p x_1 \bar{e}_n p + p \bar{e}_n y_1 p + \sum_{i=2}^m p x_i \bar{e}_n y_i p$$

Then, since  $p \in M_n(I_2/I_1)$ , by grouping terms we may write

$$p = \sum_{i=1}^{\ell+m+1} s_i \bar{e}_n t_i.$$

for  $s_i, t_i \in M_n(I_2/I_1)$ . Set

$$s = \begin{bmatrix} s_1 & s_2 & \dots & s_{\ell+m+1} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \quad \text{and} \quad t = \begin{bmatrix} t_1 & 0 & \dots & 0 \\ t_2 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ t_{\ell+m+1} & 0 & \dots & 0 \end{bmatrix}$$

Then  $s, t \in M_{n(\ell+m+1)}(I_2/I_1)$  such that

$$\begin{bmatrix} p & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} = s \bar{e}_{n(\ell+m+1)} t$$

Let  $z = s \bar{e}_{n(\ell+m+1)}$  and  $w = \bar{e}_{n(\ell+m+1)} t$ . Then  $zw = \text{diag}(p, 0, \dots, 0)$  is an idempotent. While we do not know if  $wz$  is an idempotent, we do see that  $(wz)^2 = w(zw)z = w(zw)^3z = (wz)^4$ , so that  $(wz)^2$  is an idempotent. Moreover,  $(zw)^2 = zwzw \sim wz wz = (wz)^2$  so that

$$[p] = [p^2] = [(zw)^2] = [(wz)^2]$$



in  $K_0^{\text{alg}}(I_2/I_1)$ . Since  $(wz)^2 = (\bar{e}_{n(\ell+m+1)}ts\bar{e}_{n(\ell+m+1)})^2$  we have that

$$(wz)^2 \in \bar{e}_{n(\ell+m+1)}\mathbb{M}_{n(\ell+m+1)}(I_2/I_1)\bar{e}_{n(\ell+m+1)} = \mathbb{M}_{n(\ell+m+1)}(\bar{e}(I_2/I_1)\bar{e}).$$

Since the map  $\psi : eI_2e/eI_1e \rightarrow \bar{e}(I_2/I_1)\bar{e}$  defined by  $\psi(x + eI_1e) = \bar{e}(x + I_1)\bar{e}$  is an isomorphism, the lemma follows.  $\square$

**Proposition 5.8.** *Let  $R$  be a ring and let  $e \in R$  be an idempotent such that  $ReR = R$ . Suppose every ideal of  $R$  has a countable approximate unit consisting of idempotents, and every ideal of  $eRe$  has a countable approximate unit consisting of idempotents. Then the inclusion  $\iota : eRe \rightarrow R$  induces an isomorphism from  $K_{\text{ideal}}^{\text{alg},+}(eRe)$  to  $K_{\text{ideal}}^{\text{alg},+}(R)$ .*

*Proof.* First note that since every ideal  $I$  of  $R$  has a countable approximate unit consisting of idempotents, Lemma 3.9 implies that every subquotient of  $R$  satisfies excision in algebraic  $K$ -theory. By Lemma 5.1, the map  $\beta : \text{Lat}^{\text{alg}}(eRe) \rightarrow \text{Lat}^{\text{alg}}(R)$  given by

$$\beta(I) = \text{the ideal in } R \text{ generated by } I$$

is a lattice isomorphism. We will now show that for each  $I_1, I_2 \in \text{Lat}^{\text{alg}}(R)$  with  $I_1 \subseteq I_2$ ,

$$K_n^{\text{alg}}(\iota_{I_2/I_1}) : K_n^{\text{alg}}(I_2/I_1) \rightarrow K_n^{\text{alg}}(\beta(I_2)/\beta(I_1))$$

is an isomorphism for each  $n \in \mathbb{Z}$  with  $K_0^{\text{alg}}(\iota_{I_2/I_1})$  also being an order isomorphism.

*Case 1:* Suppose  $I_2 = eRe$  and  $I_1$  is any ideal of  $eRe$ . Let  $\bar{e} := e + \beta(I_1) \in R/\beta(I_1)$ . Then  $I_1 = e\beta(I_1)e$ . We see that  $eRe/I_1 \cong \bar{e}(R/\beta(I_1))\bar{e}$ , and composing this isomorphism with the inclusion  $\bar{e}(R/\beta(I_1))\bar{e} \hookrightarrow R/\beta(I_1)$  gives  $\iota_{I_2/I_1}$ . That each  $K_n^{\text{alg}}(\iota_{I_2/I_1})$  is an isomorphism follows from Lemma 5.2, and that  $K_0^{\text{alg}}(\iota_{I_2/I_1})$  is also an order isomorphism follows from Lemma 5.7.

*Case 2:* Suppose  $I_1 = 0$  and  $I_2$  is any ideal of  $eRe$ . Note that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & I_2 & \longrightarrow & eRe & \longrightarrow & eRe/I_2 \longrightarrow 0 \\ & & \downarrow \iota_{I_2/0} & & \downarrow \iota_{eRe/0} & & \downarrow \iota_{eRe/I_2} \\ 0 & \longrightarrow & \beta(I_2) & \longrightarrow & R & \longrightarrow & R/\beta(I_2) \longrightarrow 0 \end{array}$$

commutes, and induces the following commutative diagram

$$\begin{array}{ccccccccc} K_{n+1}^{\text{alg}}(eRe) & \longrightarrow & K_{n+1}^{\text{alg}}(eRe/I_2) & \longrightarrow & K_n^{\text{alg}}(I_2) & \longrightarrow & K_n^{\text{alg}}(eRe) & \longrightarrow & K_n^{\text{alg}}(eRe/I_2) \\ \downarrow K_{n+1}^{\text{alg}}(\iota_{eRe/0}) & & \downarrow K_{n+1}^{\text{alg}}(\iota_{eRe/I_2}) & & \downarrow K_n^{\text{alg}}(\iota_{I_2/0}) & & \downarrow K_n^{\text{alg}}(\iota_{eRe/0}) & & \downarrow K_n^{\text{alg}}(\iota_{eRe/I_2}) \\ K_{n+1}^{\text{alg}}(R) & \longrightarrow & K_{n+1}^{\text{alg}}(R/\beta(I_2)) & \longrightarrow & K_n^{\text{alg}}(\beta(I_2)) & \longrightarrow & K_n^{\text{alg}}(R) & \longrightarrow & K_n^{\text{alg}}(R/\beta(I_2)) \end{array}$$

in  $K$ -theory. Since  $eIe$  and  $I$  satisfy excision in algebraic  $K$ -theory, the rows are exact. Hence, by the Five Lemma and Case 1,  $K_n^{\text{alg}}(\iota_{I_2/0})$  is an isomorphism. In addition, by Lemma 5.7  $K_0^{\text{alg}}(\iota_{I_2/0})$  is also an order isomorphism.

*Case 3:* Suppose  $I_1$  and  $I_2$  are ideals in  $eRe$  with  $I_1 \subseteq I_2$ . Then the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & I_1 & \longrightarrow & I_2 & \longrightarrow & I_2/I_1 \longrightarrow 0 \\ & & \downarrow \iota_{I_1/0} & & \downarrow \iota_{I_2/0} & & \downarrow \iota_{I_2/I_1} \\ 0 & \longrightarrow & \beta(I_1) & \longrightarrow & \beta(I_2) & \longrightarrow & \beta(I_2)/\beta(I_1) \longrightarrow 0 \end{array}$$

is commutative and induces the following commutative diagram

$$\begin{array}{ccccccccc} K_n^{\text{alg}}(I_1) & \longrightarrow & K_n^{\text{alg}}(I_2) & \longrightarrow & K_n^{\text{alg}}(I_2/I_1) & \longrightarrow & K_{n-1}^{\text{alg}}(I_1) & \longrightarrow & K_{n-1}^{\text{alg}}(I_2) \\ \downarrow K_n^{\text{alg}}(\iota_{I_1/0}) & & \downarrow K_n^{\text{alg}}(\iota_{I_2/0}) & & \downarrow K_n^{\text{alg}}(\iota_{I_2/I_1}) & & \downarrow K_{n-1}^{\text{alg}}(\iota_{I_1/0}) & & \downarrow K_{n-1}^{\text{alg}}(\iota_{I_2/0}) \\ K_n^{\text{alg}}(\beta(I_1)) & \longrightarrow & K_n^{\text{alg}}(\beta(I_2)) & \longrightarrow & K_n^{\text{alg}}(\beta(I_2)/\beta(I_1)) & \longrightarrow & K_{n-1}^{\text{alg}}(\beta(I_1)) & \longrightarrow & K_{n-1}^{\text{alg}}(\beta(I_2)) \end{array}$$

Since  $I_1$  and  $\beta(I_1)$  satisfy excision in algebraic  $K$ -theory, the rows are exact. Hence, by the Five Lemma and by Case 2, we have that  $K_n^{\text{alg}}(\iota_{I_2/I_1})$  is an isomorphism. In addition, by Lemma 5.7  $K_0^{\text{alg}}(\iota_{I_2/I_1})$  is also an order isomorphism.  $\square$

**Proposition 5.9.** *Let  $\mathfrak{A}$  be a separable  $C^*$ -algebra and let  $p \in \mathfrak{A}$  be a projection such that  $\overline{\mathfrak{A}p\mathfrak{A}} = \mathfrak{A}$ . Then the inclusion  $\iota : p\mathfrak{A}p \rightarrow \mathfrak{A}$  induces an isomorphism from  $K_{\text{ideal}}^{\text{top},+}(p\mathfrak{A}p)$  to  $K_{\text{ideal}}^{\text{top},+}(\mathfrak{A})$ .*

*Proof.* Every  $C^*$ -algebra satisfies excision in topological  $K$ -theory and by L. G. Brown [12], the natural embedding of any hereditary sub- $C^*$ -algebra of a separable  $C^*$ -algebra  $\mathfrak{B}$  that is not contained in any proper closed two-sided ideal of  $\mathfrak{B}$  induces an order isomorphism in topological  $K$ -theory. Also, note that every subquotient  $p\mathfrak{I}_2p/p\mathfrak{I}_1p$  of  $p\mathfrak{A}p$  is isomorphic to a hereditary sub- $C^*$ -algebra of  $\mathfrak{I}_2/\mathfrak{I}_1$  that is not contained in any proper closed two-sided ideal of  $\mathfrak{I}_2/\mathfrak{I}_1$ . Using these facts and using the same argument as in Proposition 5.8, we get the desired result.  $\square$

**Lemma 5.10.** *Let  $E$  be a graph with finitely many vertices and let  $F$  be a desingularization of  $E$ . Let  $\{s_e, p_v : e \in F^1, v \in F^0\} \subseteq L_{\mathbb{C}}(F) \subseteq C^*(F)$  be a generating Cuntz-Krieger  $F$ -family. Let  $p = \sum_{v \in E^0} p_v \in L_{\mathbb{C}}(F)$ . Then  $pL_{\mathbb{C}}(F)p$  is a full corner of  $L_{\mathbb{C}}(F)$ ,  $pC^*(F)p$  is a full corner of  $C^*(F)$ , and there exist isomorphisms  $\phi : L_{\mathbb{C}}(E) \rightarrow pL_{\mathbb{C}}(F)p$  and  $\psi : C^*(E) \rightarrow pC^*(F)p$  such that the diagram*

$$\begin{array}{ccccc} L_{\mathbb{C}}(E) & \xrightarrow{\phi} & pL_{\mathbb{C}}(F)p & \xhookrightarrow{\quad} & L_{\mathbb{C}}(F) \\ \downarrow \iota_E & & \downarrow \iota_F & & \downarrow \iota_F \\ C^*(E) & \xrightarrow{\psi} & pC^*(F)p & \xhookrightarrow{\quad} & C^*(F) \end{array}$$

*commutes.*

**Theorem 5.11.** *Let  $E_1$  and  $E_2$  be graphs that each have finitely many vertices and satisfy Condition (K). If there exists an isomorphism*

$$\alpha : K_{\text{ideal}}^{\text{alg},+}(L_{\mathbb{C}}(E_1)) \rightarrow K_{\text{ideal}}^{\text{alg},+}(L_{\mathbb{C}}(E_2))$$

with  $\alpha^{0, L_{\mathbb{C}}(E_1)}([1_{L_{\mathbb{C}}(E_1)}]) = [1_{L_{\mathbb{C}}(E_2)}]$ , then there exists an isomorphism

$$\beta : K_{\text{ideal}}^{\text{top},+}(C^*(E_1)) \rightarrow K_{\text{ideal}}^{\text{top},+}(C^*(E_2))$$

with  $\beta^{0, C^*(E_1)}([1_{C^*(E_1)}]) = [1_{C^*(E_2)}]$ .

*Proof.* For  $i = 1, 2$ , let  $F_i$  be a desingularization of  $E_i$ . By Lemma 5.10, for each  $i = 1, 2$  there exists an idempotent  $p_i \in L_{\mathbb{C}}(F_i)$  such that  $p_i L_{\mathbb{C}}(F_i) p_i$  is a full corner of  $L_{\mathbb{C}}(F_i)$ ,  $p_i C^*(F_i) p_i$  is a full corner of  $C^*(F_i)$ , and there exist isomorphisms  $\phi_i : L_{\mathbb{C}}(E_i) \rightarrow p_i L_{\mathbb{C}}(F_i) p_i$  and  $\psi_i : C^*(E_i) \rightarrow p_i C^*(F_i) p_i$  such that the diagram

$$\begin{array}{ccccc} L_{\mathbb{C}}(E_i) & \xrightarrow{\phi_i} & p_i L_{\mathbb{C}}(F_i) p_i & \xrightarrow{\iota^{\text{alg}}} & L_{\mathbb{C}}(F_i) \\ \downarrow \iota_{E_i} & & \downarrow \iota_{F_i} & & \downarrow \iota_{F_i} \\ C^*(E_i) & \xrightarrow{\psi_i} & p_i C^*(F_i) p_i & \xrightarrow{\iota^{\text{top}}} & C^*(F_i) \end{array}$$

commutes.

By hypothesis there is an isomorphism

$$\alpha : K_{\text{ideal}}^{\text{alg},+}(L_{\mathbb{C}}(E_1)) \rightarrow K_{\text{ideal}}^{\text{alg},+}(L_{\mathbb{C}}(E_2))$$

with  $\alpha^{0, L_{\mathbb{C}}(E_1)}([1_{L_{\mathbb{C}}(E_1)}]) = [1_{L_{\mathbb{C}}(E_2)}]$ . Since  $p_i L_{\mathbb{C}}(F_i) p_i \cong L_{\mathbb{C}}(E_i)$ , there exists an isomorphism  $\tilde{\alpha} : K_{\text{ideal}}^{\text{alg},+}(p_1 L_{\mathbb{C}}(F_1) p_1) \rightarrow K_{\text{ideal}}^{\text{alg},+}(p_2 L_{\mathbb{C}}(F_2) p_2)$  such that  $\tilde{\alpha}^{0, p_1 L_{\mathbb{C}}(F_1) p_1}$  sends  $[p_1]$  to  $[p_2]$ . By Proposition 5.8,  $\iota^{\text{alg}}$  induces an isomorphism from  $K_{\text{ideal}}^{\text{alg},+}(p_i L_{\mathbb{C}}(F_i) p_i)$  to  $K_{\text{ideal}}^{\text{alg},+}(L_{\mathbb{C}}(F_i))$ . Composing isomorphisms gives an isomorphism  $\lambda : K_{\text{ideal}}^{\text{alg},+}(L_{\mathbb{C}}(F_1)) \rightarrow K_{\text{ideal}}^{\text{alg},+}(L_{\mathbb{C}}(F_2))$  such that  $\lambda^{0, L_{\mathbb{C}}(F_1)}([p_1]) = [p_2]$ .

Since  $F_i$  for  $i = 1, 2$  is a row-finite graph, by Theorem 4.9, there exists an isomorphism  $\delta : K_{\text{ideal}}^{\text{top},+}(C^*(F_1)) \rightarrow K_{\text{ideal}}^{\text{top},+}(C^*(F_2))$  such that

$$\begin{aligned} \delta^{0, C^*(F_1)}([p_1]) &= \left( \delta^{0, C^*(F_1)} \circ \gamma_0^{F_1} \right) ([p_1]) \\ &= \left( \gamma_0^{F_2} \circ \lambda^{0, L_{\mathbb{C}}(F_1)} \right) ([p_1]) \\ &= \gamma_0^{F_2}([p_2]) \\ &= [p_2] \end{aligned}$$

By Proposition 5.9,  $\iota^{\text{top}}$  induces an isomorphism from  $K_{\text{ideal}}^{\text{top},+}(p_i C^*(F_i) p_i)$  to  $K_{\text{ideal}}^{\text{top},+}(C^*(F_i))$ . Thus, there exists an isomorphism

$$\zeta : K_{\text{ideal}}^{\text{top},+}(p_1 C^*(F_1) p_1) \rightarrow K_{\text{ideal}}^{\text{top},+}(p_2 C^*(F_2) p_2)$$

such that  $\zeta^{0, C^*(F_1)}([p_1]) = [p_2]$ . Since  $C^*(E_i) \cong p_i C^*(F_i) p_i$ , there exists an isomorphism  $\beta : K_{\text{ideal}}^{\text{top},+}(C^*(E_1)) \rightarrow K_{\text{ideal}}^{\text{top},+}(C^*(E_2))$  such that  $\beta^{0, C^*(E_1)}$  sends  $[1_{C^*(E_1)}]$  to  $[1_{C^*(E_2)}]$ . □

**Theorem 5.12.** *Let  $\mathcal{C}_1$  be a class of graphs that satisfies the following two properties:*

- (1) *Every graph in  $\mathcal{C}_1$  has finitely many vertices and satisfies Condition (K).*
- (2) *If  $E, F \in \mathcal{C}_1$  and there exists an isomorphism*

$$\alpha : K_{\text{ideal}}^{\text{top},+}(C^*(E)) \rightarrow K_{\text{ideal}}^{\text{top},+}(C^*(F))$$

*such that  $\alpha_0^{0,C^*(E)}$  sends  $[1_{C^*(E)}]$  to  $[1_{C^*(F)}]$ , then  $C^*(E) \cong C^*(F)$  (as  $*$ -algebras).*

*Then the isomorphism conjecture holds for all graphs in  $\mathcal{C}$ . In other words, if  $E, F \in \mathcal{C}$  and  $L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$  (as rings), then  $C^*(E) \cong C^*(F)$  (as  $*$ -algebras).*

*Proof.* Suppose  $E, F \in \mathcal{C}_1$  and  $L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$  (as rings). Then the ring isomorphism from  $L_{\mathbb{C}}(E)$  to  $L_{\mathbb{C}}(F)$  induces an isomorphism

$$\alpha : K_{\text{ideal}}^{\text{alg},+}(L_{\mathbb{C}}(E)) \rightarrow K_{\text{ideal}}^{\text{alg},+}(L_{\mathbb{C}}(F))$$

with  $\alpha^{0,L_{\mathbb{C}}(E)}([1_{L_{\mathbb{C}}(E)}]) = [1_{L_{\mathbb{C}}(F)}]$ . By Property (1)  $E$  and  $F$  each have finitely many vertices and satisfy Condition (K), and thus Theorem 5.11 implies there exists an isomorphism

$$\beta : K_{\text{ideal}}^{\text{top},+}(C^*(E)) \rightarrow K_{\text{ideal}}^{\text{top},+}(C^*(F))$$

with  $\beta^{0,C^*(E)}([1_{C^*(E)}]) = [1_{C^*(F)}]$ . Hence, it follows from Property (2) that  $C^*(E) \cong C^*(F)$ .  $\square$

We shall now obtain corollaries to Theorem 5.12 showing that the isomorphism conjecture for graph algebras holds for various classes of graphs where a classification up to isomorphism has been obtained for the associated  $C^*$ -algebras. Unlike the corollaries to Theorem 4.10, where we had numerous classes of graph  $C^*$ -algebras where classification up to Morita equivalence (equivalently, stable isomorphism) had been obtained, we have fewer classifications up to isomorphism. In fact, there are only three classes we are able to discuss: unital  $C^*$ -algebras of amplified graphs, unital graph  $C^*$ -algebras with exactly one ideal, and unital graph  $C^*$ -algebras that are simple.

**Corollary 5.13.** *If  $E$  and  $F$  are amplified graphs with finitely many vertices, and if  $L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$  (as rings), then  $C^*(E) \cong C^*(F)$  (as  $*$ -algebras).*

*Proof.* Let  $\mathcal{C}_1 = \{E : E \text{ is an amplified graph with finitely many vertices}\}$ . Then every graph in  $\mathcal{C}_1$  has finitely many vertices and satisfies Condition (K), so  $\mathcal{C}_1$  satisfies Property (1) of Theorem 5.12. It follows from the results of [21] that  $C^*$ -algebras of amplified graphs with finitely many vertices are classified up to isomorphism by ideal-related topological  $K$ -theory together with the positions of the units, and thus  $\mathcal{C}_1$  satisfies Property (2) of Theorem 5.12.  $\square$

**Corollary 5.14.** *If  $E$  and  $F$  are graphs with finitely many vertices and such that  $C^*(E)$  and  $C^*(F)$  have exactly one proper, nonzero ideal, and if  $L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$  (as rings), then  $C^*(E) \cong C^*(F)$  (as  $*$ -algebras).*

*Proof.* Let

$$\mathcal{C}_1 = \{E : E \text{ has finitely many vertices and } C^*(E) \text{ has exactly one ideal}\}.$$

Then every graph in  $\mathcal{C}_1$  has finitely many vertices, and since  $C^*(E)$  has finitely many ideals,  $E$  satisfies Condition (K). Thus  $\mathcal{C}_1$  satisfies Property (1) of Theorem 5.12. Recent result of Eilers, Restorff, and the first named author in [18] together with the results in [25] show that  $C^*$ -algebras of the graphs in  $\mathcal{C}_1$  are classified up to isomorphism by ideal-related topological  $K$ -theory together with the positions of the units, and thus  $\mathcal{C}_1$  satisfies Property (2) of Theorem 5.12.  $\square$

**Corollary 5.15.** *If  $E$  and  $F$  are graphs with finitely many vertices and such that  $C^*(E)$  and  $C^*(F)$  are simple, and if  $L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$  (as rings), then  $C^*(E) \cong C^*(F)$  (as  $*$ -algebras).*

*Proof.* Let

$$\mathcal{C}_1 = \{E : E \text{ is a graph with finitely many vertices and } C^*(E) \text{ is simple}\}.$$

Then every graph in  $\mathcal{C}_1$  has finitely many vertices, and since  $C^*(E)$  is simple  $E$  satisfies Condition (K). Thus  $\mathcal{C}_1$  satisfies Property (1) of Theorem 5.12. The dichotomy for simple graph  $C^*$ -algebras implies that any simple graph  $C^*$ -algebra is either AF or purely infinite, and thus either Elliott's theorem or the Kirchberg-Phillips classification theorem imply that  $C^*$ -algebras of the graphs in  $\mathcal{C}_1$  are classified up to isomorphism by their  $K$ -groups together with the position of the unit in the  $K_0$ -group. Since ideal-related  $K$ -theory reduces to the  $K$ -groups for simple  $C^*$ -algebras,  $\mathcal{C}_1$  satisfies Property (2) of Theorem 5.12.  $\square$

## 6. CLASSIFICATION OF LEAVITT PATH ALGEBRAS OF AMPLIFIED GRAPHS

In this section we use our prior results to prove a classification theorem for Leavitt path algebras. We first need the following theorem which is the analogue of the  $C^*$ -algebra fact stated in [21, Theorem 3.8].

**Theorem 6.1.** *Let  $\alpha = \alpha_1\alpha_2\cdots\alpha_n$  be a path in a graph  $E$ . Let  $F$  be the graph with vertex set  $E^0$ , edge set*

$$F^1 = E^1 \cup \{\alpha^m \mid m \in \mathbb{N}\},$$

*and range and source maps that extend those of  $E$  and have  $r_F(\alpha^m) = r_E(\alpha)$  and  $s_F(\alpha^m) = s_E(\alpha)$ . If*

$$|s_E^{-1}(s_E(\{\alpha_1\})) \cap r_E^{-1}(r_E(\{\alpha_1\}))| = \infty,$$

*then there exists  $K$ -algebra isomorphism from  $L_K(E)$  to  $L_K(F)$ .*

The proof of Theorem 6.1 is nearly identical to the proof of [21, Theorem 3.8], using [7, Theorem 3.7] in place of the gauge-invariant uniqueness theorem, and therefore we omit it.

**Definition 6.2.** Let  $E = (E^0, E^1, r_E, s_E)$  be a graph. The *amplification* of  $E$ , denoted by  $\overline{E}$ , is the graph defined by  $\overline{E}^0 := E^0$ ,

$\overline{E}^1 := \{e(v, w)^n : n \in \mathbb{N}, v, w \in E^0 \text{ and there exists an edge from } v \text{ to } w\}$ ,  
and  $s_{\overline{E}}(e(v, w)^n) := v$ , and  $r_{\overline{E}}(e(v, w)^n) := w$ .

**Definition 6.3.** Let  $E = (E^0, E^1, r_E, s_E)$  be a graph. We define the *transitive closure* of  $E$  to be the graph  $\mathfrak{t}E$  given by:

$$\mathfrak{t}E^0 := E^0,$$

$$\mathfrak{t}E^1 := E^1 \cup \{e(v, w) : \text{there is a path but no edge from } v \text{ to } w\},$$

with range and source maps that extend those of  $E$  and satisfy

$$s_{\mathfrak{t}E}(e(v, w)) := v,$$

$$r_{\mathfrak{t}E}(e(v, w)) := w.$$

**Corollary 6.4.** *If  $E$  is a graph with  $|E^0| < \infty$ , then there exists a  $K$ -algebra isomorphism  $L_K(\overline{E})$  to  $L_K(\mathfrak{t}E)$ .*

*Proof.* For any path  $\alpha$  in an amplified graph, we have

$$|s_G^{-1}(s_G(\{\alpha_1\})) \cap r_G^{-1}(r_G(\{\alpha_1\}))| = \infty,$$

so Theorem 6.1 applied a finite number of times proves the desired result.  $\square$

**Definition 6.5.** Let  $E = (E^0, E^1, r_E, s_E)$  and  $F = (F^0, F^1, r_F, s_F)$ . We say that  $E$  and  $F$  are *isomorphic*, and write  $E \cong F$ , if there exist bijections  $\alpha^0 : E^0 \rightarrow F^0$  and  $\alpha^1 : E^1 \rightarrow F^1$  such that

$$r_F(\alpha^1(e)) = \alpha^0(r_E(e)) \quad \text{and} \quad s_F(\alpha^1(e)) = \alpha^0(s_E(e)).$$

**Theorem 6.6.** [21, Theorem 5.7] *Let  $E_1$  and  $E_2$  be graphs with finitely many vertices. Then the following are equivalent.*

- (a)  $C^*(\overline{E_1}) \cong C^*(\overline{E_2})$  (as  $*$ -algebras).
- (c)  $C^*(\mathfrak{t}E_1) \cong C^*(\mathfrak{t}E_2)$  (as  $*$ -algebras).
- (d)  $\mathfrak{t}E_1 \cong \mathfrak{t}E_2$ .
- (e)  $K_{\text{ideal}}^{\text{top},+}(C^*(\overline{E_1})) \cong K_{\text{ideal}}^{\text{top},+}(C^*(\overline{E_2}))$ .

We shall now prove an analogue of [21, Theorem 5.7] for Leavitt path algebras. This result also shows that a converse to Corollary 5.13 holds.

**Theorem 6.7.** *Let  $E_1$  and  $E_2$  be graphs with finitely many vertices. Then the following are equivalent.*

- (a)  $\mathfrak{t}E_1 \cong \mathfrak{t}E_2$ .
- (b)  $L_{\mathbb{C}}(\overline{E_1}) \cong L_{\mathbb{C}}(\overline{E_2})$  (as  $\mathbb{C}$ -algebras).
- (c)  $L_{\mathbb{C}}(\overline{E_1}) \cong L_{\mathbb{C}}(\overline{E_2})$  (as rings).

- (d)  $K_{\text{ideal}}^{\text{alg},+}(L_{\mathbb{C}}(\overline{E_1})) \cong K_{\text{ideal}}^{\text{alg},+}(L_{\mathbb{C}}(\overline{E_2}))$ .
- (e)  $K_{\text{ideal}}^{\text{top},+}(C^*(\overline{E_1})) \cong K_{\text{ideal}}^{\text{top},+}(C^*(\overline{E_2}))$ .
- (f)  $C^*(\overline{E_1}) \cong C^*(\overline{E_2})$  (as  $*$ -algebras).

*Proof.* To see (a)  $\Rightarrow$  (b), suppose  $\mathfrak{t}\overline{E_1} \cong \mathfrak{t}\overline{E_2}$ . Then  $L_{\mathbb{C}}(\mathfrak{t}\overline{E_1}) \cong L_{\mathbb{C}}(\mathfrak{t}\overline{E_2})$  as  $\mathbb{C}$ -algebras. By Corollary 6.4,  $L_{\mathbb{C}}(\overline{E_i}) \cong L_{\mathbb{C}}(\mathfrak{t}\overline{E_i})$  as  $\mathbb{C}$ -algebras. Hence,  $L_{\mathbb{C}}(\overline{E_1}) \cong L_{\mathbb{C}}(\overline{E_2})$  as  $\mathbb{C}$ -algebras. Thus (b) holds.

The implications (b)  $\Rightarrow$  (c) and (c)  $\Rightarrow$  (d) are clear. To see (d)  $\Rightarrow$  (e), suppose  $K_{\text{ideal}}^{\text{alg},+}(L_{\mathbb{C}}(\overline{E_1})) \cong K_{\text{ideal}}^{\text{alg},+}(L_{\mathbb{C}}(\overline{E_2}))$ . By Lemma 5.10, Proposition 5.8, and Proposition 5.9,

$$K_{\text{ideal}}^{\text{alg},+}(L_{\mathbb{C}}(\overline{E_i})) \cong K_{\text{ideal}}^{\text{alg},+}(L_{\mathbb{C}}(F_i)) \text{ and } K_{\text{ideal}}^{\text{top},+}(C^*(\overline{E_i})) \cong K_{\text{ideal}}^{\text{top},+}(C^*(F_i)),$$

where  $F_i$  is a desingularization of  $\overline{E_i}$ , for each  $i = 1, 2$ . Therefore,

$$K_{\text{ideal}}^{\text{alg},+}(L_{\mathbb{C}}(F_1)) \cong K_{\text{ideal}}^{\text{alg},+}(L_{\mathbb{C}}(F_2)).$$

By Theorem 4.9,  $K_{\text{ideal}}^{\text{top},+}(C^*(F_1)) \cong K_{\text{ideal}}^{\text{top},+}(C^*(F_2))$ . Hence,

$$K_{\text{ideal}}^{\text{top},+}(C^*(\overline{E_1})) \cong K_{\text{ideal}}^{\text{top},+}(C^*(\overline{E_2}))$$

and (e) holds.

Lastly, (e)  $\Rightarrow$  (f) and (f)  $\Rightarrow$  (a) follow from Theorem 6.6.  $\square$

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